THE CONSTRUCTION AND STUDY OF CERTAIN IMPORTANT ALGEBRAS

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1. The Construction and Study of Certain Important Algebras.
   By Claude Chevalley.

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CONVENTIONS

Throughout these lectures, we mean by a ring a ring with unit element 1 (or 1′ as the case may be), and also by a homomorphism of such rings a homomorphism which maps unit upon unit. A will always denote a ring which is quite arbitrary in Chap. I, and assumed to be commutative in Chap. II and the subsequent chapters.

By a *module* over A, we invariably mean a unitary module. Thus a module over A is a set M such that

1) M has a structure of an additive group,
2) for every $\alpha \in A$ and $x \in M$, an element $\alpha x \in M$ called *scalar multiple* is defined and we have
   i) $\alpha(x+y)=\alpha x+\alpha y$,
   ii) $(\alpha+\beta)x=\alpha x+\beta x$,
   iii) $\alpha(\beta x)=(\alpha\beta)x$,
   iv) $1\cdot x=x$.

A map of a module over A into a module over A is called *linear*, if it is a homomorphism of the underlying additive groups which commutes with every scalar multiplication by every element of A.

An *algebra* E over A means a module over A with an associative multiplication which makes E a ring satisfying

$$\alpha(xy)=(\alpha x)y=x(\alpha y) \qquad (x, y \in E;\ \alpha \in A).$$

A homomorphism of algebras will always mean a ring homomorphism which maps unit upon unit. An ideal of an algebra means always a *two-sided ideal*. A subset S of an algebra is called *a set of generators* of E if E is the smallest subalgebra containing S and *the unit* 1 of E.

In dealing with modules or algebras over A, any element of the basic ring A is often called a *scalar*. In the case of algebras, any element of the subalgebra $A\cdot 1$ is called a scalar; a scalar clearly commutes with every element of the algebra.

CHAPTER I. GRADED ALGEBRAS.

§ 1. **Free algebras.** The first basic type of algebras we want to consider is the free algebra. Let E be an algebra over A generated by a given set of generators $(x_i)_{i\in I}$ (I: any set of indices). Let $\sigma=(i_1, \cdots, i_h)$ be a finite sequence of elements of I and put $y_\sigma = x_{i_1}\cdots x_{i_h}$. The number h is called the *length* of σ. Among the "finite sequences" we always admit the empty sequence σ_0, whose length is 0, i.e., a sequence with no term, and we put $y_{\sigma_0}=1$. We define the composition of two finite sequences $\sigma=(i_1, \cdots, i_h)$ and $\sigma'=(j_1, \cdots, j_k)$ by $\sigma\sigma'=(i_1, \cdots, i_h, j_1, \cdots, j_k)$. For σ_0, we define $\sigma_0\sigma=\sigma\sigma_0=\sigma$, i.e., σ_0 is the unit for this composition. Evidently this composition is associative: $(\sigma\sigma')\sigma''=\sigma(\sigma'\sigma'')$, and we have $y_{\sigma\sigma'}=y_\sigma y_{\sigma'}$.

THEOREM 1.1. *Every element of E is a linear combination of the y_σ's, σ running over all finite sequences of elements of I.*

PROOF. Denote by E_1 the module spanned by all the y_σ's. We shall show $E=E_1$. First we prove:

LEMMA 1.1. *E_1 is closed under multiplication.*

PROOF. Let z, z' be two elements of E_1 and put

$$z=\sum_\sigma a_\sigma y_\sigma\,, \qquad z'=\sum_\sigma a'_\sigma y_\sigma .$$

Though these two sums seem apparently infinite, we have in fact $a_\sigma=0$ and $a'_\sigma=0$ except for a finite number of σ's. Then we have

$$zz'=\sum_{\sigma,\sigma'} a_\sigma a'_{\sigma'} y_{\sigma\sigma'}\,, \qquad y_{\sigma\sigma'}\in E_1\,;$$

the sum being finite, we have $zz'\in E_1$.

Now we return to the proof of Theorem 1.1. The module E_1 is thus a subalgebra of E, and if $\sigma=(i)$ $y_\sigma=x_i$ and also $y_{\sigma_0}=1$. Therefore E_1, containing the set of generators (x_i) and 1, contains E itself, so that we obtain $E=E_1$, which proves the theorem.

DEFINITION 1.1. *If the y_σ's are linearly independent over A, then E is called a free algebra, and the set $(x_i)_{i\in I}$ is called a free system of generators of E.*

Existence and uniqueness of free algebras. We first prove the *uniqueness.* For this, we shall show a more precise condition called "universality". An algebra F over A with a system of generators $(x_i)_{\iota\in I}$ is called *universal,* if given any algebra E over A generated by a set of elements $(\xi_i)_{\iota\in I}$ indexed by the same set I, there is a unique homomorphism $\varphi: F\rightarrow E$ such that $\varphi(x_i)=\xi_i$ for all i.

THEOREM 1.2. *A free algebra F with its free system of generators is universal.*

PROOF. By definition, the set $\{y_\sigma=x_{i_1}\cdots x_{i_h}\}$ forms a base of F as a *module* over A. Thus there is a *linear* mapping $\varphi: F\rightarrow E$ such that

$$(1) \qquad \varphi(y_\sigma)=\xi_{i_1}\cdots\xi_{i_h} \qquad \text{for every } \sigma=(i_1,\cdots,i_h)\,.$$

If $\sigma=(i_1, \ \ ,i_h)$, $\sigma'=(j_1,\cdots,j_k)$ are two finite sequences of I, we have

$$(2) \qquad \varphi(y_\sigma y_{\sigma'})=\varphi(y_{\sigma\sigma'})=\xi_{i_1}\cdots\xi_{i_h}\xi_{j_1}\cdots\xi_{j_k}=\varphi(y_\sigma)\varphi(y_{\sigma'})\,.$$

This proves that φ is not only linear, but also a homomorphism $F\rightarrow E$. Especially putting $\sigma=(i)$ resp. $\sigma=\sigma_0$, we have $\varphi(x_i)=\xi_i$ and $\varphi(1)=1$, which prove our assertion.

Remark that, in general, any homomorphism φ is uniquely determined when the values $\varphi(x_i)$ on a set of generators (x_i) are given.

COROLLARY. *The free algebra generated by $(x_i)_{\iota\in I}$ is unique under isomorphism. More precisely, let F, F' be two free algebras with free systems of generators $(x_i)_{i\in I}$, $(x'_{i'})_{i'\in I'}$ respectively, and let I and I' be equipotent. Then F and F' are isomorphic.*

PROOF. We may assume that $I=I'$. By Theorem 1.2, we have two homomorphisms

$$\varphi : F\rightarrow F' \qquad \text{such that } \varphi(x_i)=x'_i$$

and

$$\varphi' : F'\rightarrow F \qquad \text{such that } \varphi'(x'_i)=x_i$$

The composite mapping $\varphi'\circ\varphi$[1]: $F\rightarrow F'\rightarrow F$ maps each x_i to itself,

1) $\varphi'\circ\varphi$ is defined by $\varphi'\circ\varphi(x)=\varphi'(\varphi(x))$.

and by the uniqueness of homomorphism, $\varphi' \circ \varphi$ must be the identity in F. Similarly $\varphi \circ \varphi'$ is the identity in F'. Therefore φ is an isomorphism and $\varphi' = \varphi^{-1}$ which proves that F and F' are isomorphic to each other.

Now we shall prove the *existence* of a free algebra, having any given set $(x_i)_{i \in I}$ as its free system of generators. Let Σ be the set of all finite sequences of elements of I. From the theory of linear algebra, we may assume that there exists a module M over A with a base equipotent to Σ. Let $(y_\sigma)_{\sigma \in \Sigma}$ be the base of M; we introduce a structure of algebra into M. For this, we have only to define an associative multiplication for the elements of the base. We define it by

$$y_\sigma y_{\sigma'} = y_{\sigma\sigma'} .$$

Since the composition in Σ is associative, we have the associativity: $(y_\sigma y_{\sigma'}) y_{\sigma''} = y_\sigma (y_{\sigma'} y_{\sigma''})$. M is now a free algebra over A having the free system of generators $(x_i)_{i \in I}$.

§ 2. **Graded algebras.** Let F be the free algebra with the free system of generators $(x_i)_{i \in I}$, and put $y_\sigma = x_{i_1} \cdots x_{i_h}$ $(\sigma = (i_1, \cdots, i_h))$. We shall classify the elements y_σ by the length of σ.

Let F_h be the module spanned by the y_σ's, σ being of length h. Then F is the direct sum of $F_0, F_1, F_2, \cdots$ as a module:

$$F = F_0 + F_1 + F_2 + \cdots + F_h + \cdots \tag{1}$$

and evidently

$$F_h \cdot F_{h'} \subset F_{h+h'} , \tag{2}$$

because the length of the composite $\sigma\sigma'$ of σ and σ' is equal to the sum of the lengths of σ and σ'.

The free algebra $F = F_0 + F_1 + \cdots + F_h + \cdots$ is a typical example of the following general notion of *graded algebras*.

DEFINITION 1.2. *Let Γ be an additive group. A Γ-graded algebra is an algebra E which is given together with a direct sum decomposition as a module*

$$E = \sum_{\gamma \in \Gamma} E_\gamma \tag{3}$$

where the E_γ's are submodules of E, in such a way that

(4) $E_\gamma \cdot E_{\gamma'} \subset E_{\gamma+\gamma'}$, *i.e.*, $x \in E_\gamma$ *and* $x' \in E_{\gamma'}$ *imply* $xx' \in E_{\gamma+\gamma'}$.

By a homomorphism of Γ-graded algebra $E = \sum_{\gamma \in \Gamma} E_\gamma$ into another Γ-graded algebra $E' = \sum_{\gamma \in \Gamma} E'_\gamma$ is meant a homomorphism $\varphi: E \to E'$ of the algebras such that $\varphi(E_\gamma) \subset E'_\gamma$.

In a Γ-graded algebra $E = \sum E_\gamma$ an element belonging to E_γ is called *homogeneous* of degree γ. The zero element 0 of E is homogeneous of any degree, but each element of E other than 0 is homogeneous of at most one degree $\gamma \in \Gamma$. Any element x of E is uniquely decomposed into the sum of homogeneous elements

(5) $$x = \sum_{\gamma \in \Gamma} x_\gamma, \qquad x_\gamma \in E_\gamma,$$

where the x_γ's are 0 except for a finite number of γ's. Each x_γ in (5) is called the γ-*component* of x.

LEMMA 1.2. *The unit* 1 *is always homogeneous of degree θ (θ: zero element of Γ).*

PROOF. Decompose 1 into the sum of its homogeneous components:

$$1 = \sum_{\gamma \in \Gamma} e_\gamma, \qquad e_\gamma \in E_\gamma.$$

If $x_\beta \in E$ is homogeneous of degree $\beta \in \Gamma$, then we have

$$E_\beta \ni x_\beta = x_\beta \cdot 1 = \sum_\gamma x_\beta \cdot e_\gamma.$$

Since $x_\beta \cdot e_\gamma \in E_{\beta+\gamma}$, we must have $x_\beta \cdot e_\theta = x_\beta$ and $x_\beta \cdot e_\gamma = 0$ for all $\gamma \neq \theta$. This implies that e_θ is a right unit element for all homogeneous elements, and accordingly for all elements $x = \sum x_\gamma$ in E. Thus $e_\theta = 1$, and our assertion is proved.

COROLLARY. *Scalars are homogeneous of degree θ (θ: zero element of Γ).*

Among others, the following two special types of Γ-gradations are of much importance:

i) Γ-gradations where $\Gamma = Z$ is the additive group of integers. In this case, we say simply "graded" instead of "Z-graded".

ii) Γ-gradations where Γ is the group with two elements 0 and 1. In this case we write $E=E_++E_-$ in place of $E=E_0+E_1$, and E is called *semi-graded.*

A free algebra $F=F_0+F_1+\cdots+F_h+\cdots$ can be considered as a graded algebra with $F_h=\{0\}$ for all $h<0$.

REMARK. A Γ-graded algebra is not a special kind of algebras. In fact, any algebra may be considered as a Γ-graded algebra with degree θ for every element.

Homogeneous subalgebras.

DEFINITION 1.3. *A submodule M of a Γ-graded algebra $E=\sum E_\gamma$ is said to be homogeneous if the homogeneous components of any element of M still belong to M. This is equivalent to the condition that $M=\sum_\gamma (M \cap E_\gamma)$.*

THEOREM 1.3. *If a submodule M or an ideal $\mathfrak{A}$ of a Γ-graded algebra E is generated by*[2] *homogeneous elements, then it is homogeneous.*

PROOF. Let M be a submodule of E spanned by a set S of homogeneous elements and let M' be the set of elements of M whose homogeneous components belong to M. It is evident that $S\subset M'\subset M$, since S consists of homogeneous elements. We shall show that M' is a submodule. If $x=\sum x_\gamma$ and $x'=\sum x'_\gamma$ are in M', then $x\pm x'=\sum (x_\gamma \pm x'_\gamma)$, and $x_\gamma \pm x'_\gamma \in M$, so that we have $x\pm x' \in M'$. Also for $\alpha \in A$, we have similarly $\alpha x \in M'$. Thus M' being a submodule containing the generators S, we have $M' \supset M$, and so $M=M'$, which proves that M is homogeneous.

For the case of ideals, we take the ideal $\mathfrak{A}$ generated by a set S of homogeneous elements. $\mathfrak{A}$ is spanned, *as a module*, by all elements of the form xsy, where $x\in E$, $s\in S$ and $y\in E$. Putting $x=\sum x_\gamma$, $y=\sum y_\beta$, we have

2) The word "generated by" has somewhat different meaning for the cases of submodules and of ideals. In the former case, a submodule M is generated by S if every element of M is a linear combination of the elements of S, while in the latter case, an ideal $\mathfrak{A}$ is generated by S if $\mathfrak{A}$ is the smallest ideal containing the set S.

$$x\,s\,y=(\sum_{\gamma} x_{\gamma})s(\sum_{\beta} y_{\beta})=\sum_{\gamma,\beta} x_{\gamma}\,s\,y_{\beta}$$

and since $(x_{\gamma}\,s\,y_{\beta})$ is homogeneous, $\mathfrak{A}$ is also spanned by the elements $x_{\gamma}\,s\,y_{\beta}$ which are homogeneous. Thus $\mathfrak{A}$, being generated as a module by homogeneous elements, is homogeneous as was seen above.

Let $E=\sum E_{\gamma}$ be a Γ-graded algebra and $\mathfrak{A}$ a homogeneous ideal in E. We have the direct sum decomposition of $\mathfrak{A}$ into its homogeneous parts:

$$\mathfrak{A}=\sum_{\gamma}\mathfrak{A}_{\gamma}, \qquad \mathfrak{A}_{\gamma}=\mathfrak{A}\cap E_{\gamma}.$$

The quotient algebra $E/\mathfrak{A}$ has also the structure of *Γ-graded algebra*, because $E/\mathfrak{A}=\sum_{\gamma}(E_{\gamma}/\mathfrak{A}_{\gamma})$ (direct sum of submodules) and $(E_{\gamma}/\mathfrak{A}_{\gamma})\cdot(E_{\gamma'}/\mathfrak{A}_{\gamma'})\subset E_{\gamma+\gamma'}/\mathfrak{A}_{\gamma+\gamma'}$. Therefore $E/\mathfrak{A}$ is a Γ-graded algebra and $\sum_{\gamma}(E_{\gamma}/\mathfrak{A}_{\gamma})$ gives its homogeneous decomposition. The canonical homomorphism $\psi: E\to E/\mathfrak{A}$ is a homomorphism not only of algebras, but also of *Γ-graded algebras*.

§3. **Homogeneous linear mappings.**[3] Let E, E' be two Γ-graded algebras over the same ring A, and let λ be a linear mapping of E into E', i.e., a mapping $\lambda: E\to E'$ such that

$$\lambda(x+y)=\lambda(x)+\lambda(y), \quad \lambda(\alpha x)=\alpha\lambda(x)$$

$$\text{for every } x,y\in E; \quad \alpha\in A.$$

Definition 1.4. *Let ν be any element of Γ; λ is called homogeneous of degree ν if $\lambda(E_{\gamma})\subset E'_{\gamma+\nu}$ for all $\gamma\in\Gamma$.*

Evidently, if $\lambda: E\to E'$ is homogeneous of degree ν and $\lambda': E'\to E''$ is homogeneous of degree ν', then $\lambda'\circ\lambda$ is homogeneous of degree $\nu+\nu'$.

A linear mapping $\lambda: E\to E'$ *can not always be decomposed* into a finite sum of homogeneous mappings as can be shown by a counter-example. But if the decomposition is *possible*, it is *unique*; it is sufficient to prove the following:

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3) This notion can be defined not only for graded algebras, but also for "graded modules". But we shall restrict ourselves only to the case of graded algebras, because we use it only in this case.

LEMMA 1.3. *Let $\{\lambda_\nu\}_{\nu\in\Gamma}$ be a family of linear mappings $E\to E'$, in which each λ_ν is homogeneous of degree ν. If $\sum_\nu \lambda_\nu=0$ and $\lambda_\nu(x)=0$ (x: any element in E) except for a finite number of $\nu\in\Gamma$, then $\lambda_\nu=0$ for all $\nu\in\Gamma$.*

PROOF. For an element x_γ of E_γ, we have $\sum_\nu \lambda_\nu(x_\gamma)=0$, but since $\lambda_\nu(x_\gamma)\in E'_{\gamma+\nu}$ for each $\nu\in\Gamma$, we have $\lambda_\nu(x_\gamma)=0$ for all $\nu\in\Gamma$. For an arbitrary $x\in E$, let $x=\sum x_\gamma$ be the homogeneous decomposition of x, then $\lambda_\nu(x)=\sum_\gamma \lambda_\nu(x_\gamma)=0$, which proves that $\lambda_\nu=0$ ($\nu\in\Gamma$).

§ 4. **Associated gradations and the main involution.** Let Γ, $\tilde{\Gamma}$ be additive groups and let a homomorphism $\tau:\Gamma\to\tilde{\Gamma}$ be given. To any Γ-graded algebra $E=\sum_{\gamma\in\Gamma} E_\gamma$, we associate the following $\tilde{\Gamma}$-gradation of E. For each $\tilde{\gamma}\in\tilde{\Gamma}$, put

$$E_{\tilde{\gamma}}=\sum_{\gamma\in\tau^{-1}(\tilde{\gamma})} E_\gamma \quad (E_{\tilde{\gamma}}=\{0\} \text{ if } \tau^{-1}(\tilde{\gamma}) \text{ is empty}).$$

Then obviously $E=\sum_{\tilde{\gamma}\in\tilde{\Gamma}} E_{\tilde{\gamma}}$ and $E_{\tilde{\gamma}}\cdot E_{\tilde{\gamma}'}\subset E_{\tilde{\gamma}+\tilde{\gamma}'}$. In this way $E=\sum E_{\tilde{\gamma}}$ can be considered as a $\tilde{\Gamma}$-graded algebra.

DEFINITION 1.5. *The $\tilde{\Gamma}$-gradation $E=\sum_{\tilde{\gamma}\in\tilde{\Gamma}} E_{\tilde{\gamma}}$ is called the associated $\tilde{\Gamma}$-gradation of E, associated to the Γ-gradation $E=\sum_{\gamma\in\Gamma} E_\gamma$ (with respect to τ).*

We shall write E^τ instead of E if it is taken with the associated $\tilde{\Gamma}$-gradation rather than with the original Γ-gradation. Obviously, we have the

LEMMA 1.4. *Every homogeneous element, every homogeneous submodule, and every homogeneous ideal in E are also homogeneous in E^τ.*

In the special case where $\tilde{\Gamma}$ is the group consisting of two elements 0 and 1, and where τ is onto, we write $E^s=E^s_+ + E^s_-$ instead of $E^\tau=E_0+E_1$, and we call it the *associated semi-graded algebra* of E. In that case, the kernel $\tau^{-1}(0)\subset\Gamma$ is denoted by Γ_+, which is a subgroup of index 2, while $\tau^{-1}(1)\subset\Gamma$ is denoted by Γ_-, which is a coset of Γ by Γ_+ other than Γ_+. Remark that every subgroup of Γ of index 2 can be preassigned as Γ_+ in some unique associated semi-gradation. It may happen that Γ has a unique subgroup of index 2. If it is the case, then reference to the map τ can be omitted without any ambiguity. For example, to every graded (i.e., Z-graded) algebra $E=\sum_{h:\text{integer}} E_h$ is associated a unique semi-graded algebra $E^s=E^s_+ + E^s_-$, where $E^s_+=\sum_{h:\text{even}} E_h$, $E^s_-=\sum_{h:\text{odd}} E_h$. Clearly, if E is a semi-graded algebra, then its associated semi-gradation is identical with the original semi-gradation.

Main involution. Fixing a subgroup $\Gamma_+\subset\Gamma$ of index 2, let $E=\sum_{\gamma\in\Gamma} E_\gamma$ be a Γ-graded algebra, and let $E^s=E^s_+ + E^s_-$ be the associated semi-gradation of E. Every element $x\in E$ can be decomposed uniquely into the sum of its E^s_+-component x_+ and its E^s_--component x_- : $x=x_+ + x_-$. If we define a map $J: E\rightarrow E$ by

$$J(x)=x_+ - x_- \quad (x=x_+ + x_- \in E),$$

then J is one-to-one and linear, preserves the degree in the Γ-gradation of E, maps unit upon unit, and is an involution (i.e., $J\circ J=$ identity). Moreover, J preserves the multiplication. In fact, let $x=x_+ + x_-$, $y=y_+ + y_-$ $(x_+, y_+ \in E^s_+ ;\ x_-, y_- \in E^s_-)$. Then $(xy)_+=x_+y_+ + x_-y_-$, $(xy)_-=x_-y_+ + x_+y_-$, and so we have

$$\begin{aligned}J(xy)&=(x_+y_+ + x_-y_-)-(x_-y_+ + x_+y_-)\\ &=(x_+ - x_-)(y_+ - y_-)=J(x)J(y).\end{aligned}$$

Therefore, J is an involutive automorphism of the Γ-graded algebra E, which we call the *main involution* of E.

For convenience' sake, we define the symbolical power $J^\nu (\nu\in\Gamma)$ of the main involution as follows:

$$J^\nu=\begin{cases} J & \text{if } \nu\in\Gamma_- \\ \text{identity} & \text{if } \nu\in\Gamma_+ . \end{cases}$$

Also we define the power $(-1)^\nu (\nu\in\Gamma)$ of the scalar (-1) of A as follows:

$$(-1)^\nu=\begin{cases} -1 & \text{if } \nu\in\Gamma_- \\ 1 & \text{if } \nu\in\Gamma_+ . \end{cases}$$

Then we have, just as in the case of usual powers, the following identities:

i) $J^{\nu}\circ J^{\nu'}=J^{\nu+\nu'}$

ii) $(-1)^{\nu}(-1)^{\nu'}=(-1)^{\nu+\nu'}$

iii) $(J^{\nu})^{\nu'}=(J^{\nu'})^{\nu}$

iv) $((-1)^{\nu})^{\nu'}=((-1)^{\nu'})^{\nu}$

We shall denote iii) and iv) respectively by $J^{\nu\nu'}$ and by $(-1)^{\nu\nu'}$ for the sake of simplicity, though no product is defined, in general in Γ. Any power of the identity map is understood to be the identity map, and any power of 1 is understood to be 1.

If $x=\sum_{\gamma\in\Gamma}x_{\gamma}\quad(x_{\gamma}\in E_{\gamma})$, then we can write

v) $J(x)=\sum_{\gamma\in\Gamma}(-1)^{\gamma}x_{\gamma}$.

If $\Gamma=Z$, the additive group of integers, then these definitions agree with the usual definitions of powers of an automorphism, or of an element of an algebra.

§5. **Derivations.** The definition of derivations in a graded algebra given here is somewhat different from the conventional definition of the derivations in the ordinary algebraic systems. In the sequel, when we speak of derivations, we understand that a fixed subgroup $\Gamma_{+}\subset\Gamma$ of index 2 is given.

Now, let E, E' be two Γ-graded algebras over A and let φ be a homomorphism of E into E'.

DEFINITION 1.6. *A φ-derivation D of E into E' means a linear mapping $D:E\to E'$, homogeneous of some given degree $\nu\in\Gamma$, such that for every $x,y\in E$,*

$$D(xy)=D(x)\varphi(y)+\varphi(J^{\nu}x)D(y), \tag{1}$$

where J^{ν} is the power of the main involution defined above.

In the case where $E=E'$ and φ is the identity, D is called simply a "*derivation*". Therefore a derivation D of E is a homogeneous linear mapping of degree ν, such that

$$D(xy)=D(x)y+(J^{\nu}x)D(y)\quad\text{for } x,y\in E \tag{2}$$

If $\Gamma=Z$, the additive group of integers, (2) is written by

(2′) $$D(xy)=D(x)y+(-1)^{h\nu}x\,D(y) \text{ for } x\in E_h,\ y\in E.$$

If the elements of E are all of degree θ (θ: zero element of Γ), then D must be of degree θ, and (2) reduces to

(3) $$D(xy)=D(x)y+x\,D(y),$$

which coincides with the ordinary definition of derivation. Also, when ν belong to Γ_+ (2) reduces to (3), while if ν belong to Γ_- and $x\in E_-^s$, (2) reduces to

(4) $$D(xy)=D(x)y-x\,D(y).$$

A linear mapping satisfying (4) is sometimes called "anti-derivation", but we do not use this terminology in these lectures.

The formula (1) can be written in another form. Denote by L_x the operation of the left multiplication by x: $L_x y=xy$. Then (1) is equivalent to

(5) $$D\circ L_x=L_{D(x)}\circ\varphi+L_{\varphi(J^\nu x)}\circ D.$$

In the case where $E=E'$, and φ is the identity,

(6) $$D\circ L_x=L_{D(x)}+L_{J^\nu x}\circ D.$$

Remark that (5) and (6) do not contain the "parameter" y.

LEMMA 1.5. *For every φ-derivation D, we have $D(1)=0$.*

PROOF. Substituting $x=y=1$ in (1), we get

$$D(1)=D(1\cdot 1)=D(1)\varphi(1)+\varphi(J^\nu 1)\,D(1),$$

and since $J^\nu 1=1$, $\varphi(1)=1$, we obtain $D(1)=D(1)+D(1)$, which proves $D(1)=0$.

Evidently, if D and D' are φ-derivations of the same degree, $D\pm D'$ is again a φ-derivation. Also we have

LEMMA 1.6. *If $\varphi: E\to E'$ and $\varphi': E'\to E''$ are homomorphisms and if D, D' are a φ-derivation of E into E' and a φ'-derivation of E' into E'' respectively, then $\varphi'\circ D$ and $D'\circ\varphi$ are $(\varphi'\circ\varphi)$-derivations of $E\to E''$.*

PROOF. We have only to check the condition (1). By direct calculation we have

$$(\varphi' \circ D)(xy)=\varphi'(D(x))\ \varphi'(\varphi(y))+\varphi'(\varphi(J^{\nu} x))\varphi'(D(y))$$

and

$$(D' \circ \varphi)(xy)=D'(\varphi(x))\varphi'(\varphi(y))+\varphi'(\varphi(J^{\nu'}x))D'(\varphi(y)),$$

and since $\varphi' \circ D$ and $D' \circ \varphi$ are of degrees ν and ν' respectively, we have our assertion.

THEOREM 1.4. *Let D be a φ-derivation of E into E', F a homogeneous subalgebra of E, S a set of homogeneous generators of F, and let F' be a homogeneous subalgebra of E'. Then if $D(S) \subset F'$ and $\varphi(S) \subset F'$, we have $D(F) \subset F'$ and $\varphi(F) \subset F'$.*

PROOF. The latter inclusion is evident, because φ is a homomorphism. The former is proved as follows. Let F_1 be the set of elements $x \in F$ such that $D(x) \in F'$. It is evident that F_1 is closed under addition and scalar multiplication. Also if $D(x) \in F'$ and $x=\sum x_\gamma$, then the $D(x_\gamma)$'s are the homogeneous components of $D(x)$ and $D(x_\gamma) \in F'$, so we obtain $x_\gamma \in F_1$. Therefore F_1 is a *homogeneous submodule* of F, so that $x \in F_1$ implies $J^\nu x \in F_1$. Now for $x, y \in F_1$, we have

$$D(xy)=D(x)\varphi(y)+\varphi(J^\nu x)D(y),$$

and since $D(x)$, $\varphi(y)$, $\varphi(J^\nu x)$, $D(y)$ all belong to F', we have $xy \in F_1$, which proves that F_1 is a subalgebra containing S. S being the set of generators of F, we have $F \subset F_1$, which proves $D(F) \subset F'$.

COROLLARY 1. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be homogeneous ideals of E and E' respectively, and S be a set of homogeneous generators of $\mathfrak{A}$. If $D(S) \subset \mathfrak{A}'$, $\varphi(S) \subset \mathfrak{A}'$, we have $D(\mathfrak{A}) \subset \mathfrak{A}'$, and $\varphi(\mathfrak{A}) \subset \mathfrak{A}'$.*

PROOF. Again the latter inclusion is evident. The former is proved in a similar manner as before, showing that the set

$$\mathfrak{A}_1=\{x \mid x \in \mathfrak{A},\quad D(x) \in \mathfrak{A}'\}$$

is a homogeneous ideal.

COROLLARY 2. *Let F, S be as before. If $D(S)=\{0\}$, then $D(F)=\{0\}$.*[4]

PROOF. In a similar manner as in the proof of Theorem 1.4, we can show that

4) Remark that this assertion holds without any assumption on φ.

$$F_2=\{x \mid x \in F, \quad D(x)=0\}$$

is a homogeneous subalgebra, which proves $F \subset F_2$.

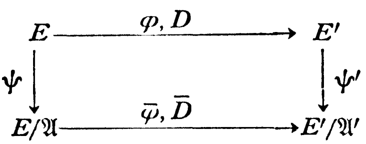
COROLLARY 3. *Let F, S be as before. If two φ-derivations D, D' coincide with each other on S, then they coincide on F.*

PROOF. From this assumption, D and D' are of the same degree. Then apply Corollary 2 to the derivation $D-D'$.

It follows from this corollary that a derivation D is completely determined if its values on the elements of a set of generators are given.

THEOREM 1.5. *Let E, E' be Γ-graded algebras, φ a homomorphism $E \to E'$, and D a φ-derivation of $E \to E'$. Also let $\mathfrak{A}$ and $\mathfrak{A}'$ be homogeneous ideals in E and E' respectively such that $D(\mathfrak{A}) \subset \mathfrak{A}'$, and $\varphi(\mathfrak{A}) \subset \mathfrak{A}'$. Under these assumptions, the induced mapping $\bar{D}$ $E/\mathfrak{A} \to E'/\mathfrak{A}'$ obtained from D is a $\bar{\varphi}$-derivation, where $\bar{\varphi}$ means the induced homomorphism $E/\mathfrak{A} \to E'/\mathfrak{A}'$ obtained from φ.*

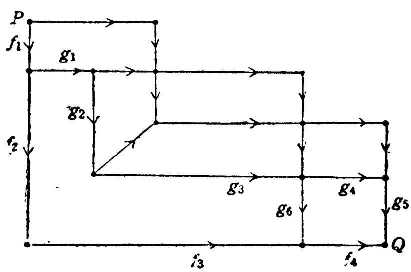
If we use the "commutative diagram"[5] the map D and $\bar{\varphi}$ are represented as follows:



where ψ and ψ' are the canonical mappings.

---

5) In a diagram, let every vertex represent a set, and let each oriented edge represent a mapping. A directed path in a diagram represent a mapping which is the composition of successive mappings assigned to its edges. If, for any two vertices, any two directed paths connecting them give the same mapping, then the diagram is said to be *commutative*. For example in Fig. 1, for the vertices P and Q and the paths as in it, the commutativity means $f_4 \circ f_3 \circ f_2 \circ f_1(x) = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ f_1(x) = f_4 \circ g_6 \circ g_3 \circ g_2 \circ g_1 \circ f_1(x) = \cdots$ for every $x \in P$.



PROOF. From the theory of mappings of modules, it is easy to see that $\bar{D}$ is a linear mapping which makes the diagram commutative. The other conditions ($\bar{D}$ being homogeneous and satisfying (1)) are proved by direct calculation from the definitions.

$\bar{D}$ is called the *derivation deduced from D* by going over to the quotient algebra $E/\mathfrak{A}$.

Hereafter to the end of this paragraph, we assume that $E=E'$ and φ is the identity.

THEOREM 1.6. *Let D, D' be two derivations of E of degrees ν and ν' respectively. Then*

$$\Delta=DD'-(-1)^{\nu\nu'}D'D \tag{7}$$

is again a derivation.[6]

PROOF. It is evident that Δ is linear and homogeneous of degree $\nu+\nu'$. We have only to check the condition (5) (equivalent to (1)). For D and D' we have by (5)

$$DL_x=L_{Dx}+L_{J^\nu x}D\,, \quad D'L_x=L_{D'x}+L_{J^{\nu'}x}D'\,.$$

Then

$$DD'L_x=DL_{D'x}+DL_{J^{\nu'}x}D'=L_{DD'x}+L_{J^\nu D'x}D+L_{DJ^{\nu'}x}D'+L_{J^{\nu+\nu'}x}DD'\,,$$

$$D'DL_x=D'L_{Dx}+D'L_{J^\nu x}D=L_{D'Dx}+L_{J^{\nu'}Dx}D'+L_{D'J^\nu x}D+L_{J^{\nu+\nu'}x}D'D\,,$$

and then

$$\Delta L_x=[DD'-(-1)^{\nu\nu'}D'D]L_x=L_{\Delta x}+L_{J^{\nu+\nu'}x}\Delta+L_{\Theta x}D'+L_{\Theta' x}D$$

where

$$\Theta=DJ^{\nu'}-(-1)^{\nu\nu'}J^{\nu'}D \text{ and } \Theta'=J^\nu D'-(-1)^{\nu\nu'}D'J^\nu\,.$$

Now it is sufficient to prove that $\Theta=\Theta'=0$, i.e.,

$$DJ^{\nu'}=(-1)^{\nu\nu'}J^{\nu'}D \text{ and } J^\nu D'=(-1)^{\nu\nu'}D'J^\nu\,. \tag{8}$$

But the former one is obtained from the latter by exchanging D and D', so we show the latter one. For a homogeneous element x of degree γ, $D'x$ is homogeneous of degree $\gamma+\nu'$, and then

$$J^\nu D'x=(-1)^{\nu(\gamma+\nu')}D'x=(-1)^{\nu\nu'}D'(-1)^{\gamma\nu}x=(-1)^{\nu\nu'}D'J^\nu x$$

---

6) We omit the symbol $\circ$ in the composition of mapping for the sake of simplicity.

which proves (8). Thus our proof is completed.

COROLLRRY 1. *If one of* ν *and* ν' *is in* Γ_+, *and in particular when* $\nu=\nu'=\theta$, *then*

$$[D, D']=DD'-D'D$$

is again a derivation. If both ν *and* ν' *are in* Γ_-, *then*

$$DD'+D'D$$

is a derivation.

COROLLARY 2. *If* D *is a derivation of degree* $\nu\in\Gamma_-$ *then* D^2 *is also a derivation of degree* $2\nu\in\Gamma_+$.

PROOF. If we put $D=D'$ in the last part in Corollary 1, we have $2D^2$ as a derivation, and the constant coefficient 2 may be omitted, provided that A is a field of characteristic other than 2.

However, we shall prove this assertion directly as follows. The characteristic property that D is a derivation of some degree ν in Γ_- is

$$DL_x=L_{Dx}+L_{Jx}D. \tag{9}$$

Since D^2 is of degree 2ν in Γ_+, we have

$$D^2L_x=DL_{Dx}+DL_{Jx}D=L_{D^2x}+L_{DJx}D+L_{JDx}D+L_{JJx}D^2.$$

But since D is of degree $\nu\in\Gamma_-$, we have $JD=-DJ$ from (8), and then

$$D^2L_x=L_{D^2x}+L_xD^2,$$

which means that D^2 is a derivation of degree $2\nu\in\Gamma_+$.

§ 6. **Existence of derivations in free algebras.** Let F be the free algebra with free system of generators $(x_i)_{i\in I}$, over a ring A. F is so graded that x_i are of degree 1. Let E be a graded algebra over A and φ a homomorphism $F\to E$.

THEOREM 1.7. *Assume that for each* $i\in I$, *a homogeneous element* $y_i\in E$ *of degree* $\nu+1$ *is preassigned arbitrarily, where* ν *is a fixed integer. Then there exists one and only one* φ*-derivation* D *of* F *into* E, *which is of degree* ν *and satisfies* $D(x_i)=y_i$.

PROOF. The uniqueness follows from Corollary 3 to Theorem 1.4. So we shall prove the existence. By Theorem 1.1, the elements $p_\sigma = x_{i_1}\cdots x_{i_h}$ form a base of F where $\sigma=(i_1,\cdots,i_h)$ runs over the set Σ consisting of all finite sequences taken from I. We shall define $\delta(p_\sigma)\in E$ by induction of the length of σ. First we put

$$(1) \qquad \delta(p_{\sigma_0})=\delta(1)=0$$

for the empty sequence σ_0. If $\delta(p_\sigma)$ has already been defined for every σ with length less than h, we set

$$(2) \qquad \delta(x_{i_1}\cdots x_{i_h})=\delta(x_{i_1}\cdots x_{i_{h-1}})\varphi(x_{i_h})+\varphi(J^\nu(x_{i_1}\cdots x_{i_{h-1}}))y_{i_h}.$$

In the case where $h=1$, we have $\delta(x_i)=y_i$. From the definition, $\delta(p_\sigma)$ is homogeneous of degree $h+\nu$ if σ has the length h. For, if $h=1$, $\delta(x_i)=y_i$ is of degree $\nu+1$ by assumption, and if this property has already been proved up to $h-1$, the degrees of the terms on the right hand side in (2) are $(h-1+\nu)+1$ and $(h-1)+(\nu+1)$ respectively, which are both equal to $h+\nu$. Hence $\delta(p_\sigma)$ is of degree $h+\nu$.

Now we define a linear mapping $D: F\to E$ such that $D(p_\sigma)=\delta(p_\sigma)$ for all $\sigma\in\Sigma$. Since (p_σ) forms a base of F, such D always exists and is determined uniquely. Evidently D is linear and homogeneous of degree ν. Next we shall show the condition

$$(3) \qquad D(uv)=D(u)\varphi(v)+\varphi(J^\nu u)D(v) \quad (u, v\in F).$$

We first remark that

$$D(p_\sigma x_i)=D(p_\sigma)\varphi(x_i)+\varphi(J^\nu p_\sigma)D(x_i)$$

holds by (2), and then forming a linear combination of (p_σ), we obtain by linearity of D,

$$(4) \qquad D(ux_i)=D(u)\varphi(x_i)+\varphi(J^\nu u)D(x_i).$$

Now we denote by F_1 the set of all elements v of F which satisfy the condition (3) for all $u\in F$. From (4), we have $x_i\in F_1$ and also $1\in F_1$, for if $v=1$, (3) reduces to a trivial relation $D(u)=D(u)$. We shall prove that $v\in F_1$ implies $vx_i\in F_1$. In fact, substituting uv in (4), we have

$$\begin{aligned} D(uvx_i) &= D(uv)\varphi(x_i)+\varphi(J^\nu(uv))D(x_i) \\ &= D(u)\varphi(v)\varphi(x_i)+\varphi(J^\nu u)D(v)\varphi(x_i) \\ &\qquad +\varphi(J^\nu u)\varphi(J^\nu v)D(x_i) \quad \text{(since } v\in F_1\text{)} \\ &= D(u)\varphi(vx_i)+\varphi(J^\nu u)[D(v)\varphi(x_i)+\varphi(J^\nu v)D(x_i)] \\ &= D(u)\varphi(vx_i)+\varphi(J^\nu u)D(vx_i) \qquad \text{(again by (4))}. \end{aligned}$$

which proves our assertion. Therefore beginning with $x_{i_1}\in F_1$ and repeating this process, we have $p_\sigma \in F_1$ for every $\sigma=(i_1,\cdots,i_h)$. Then by the linearity of D, we have finally that all the elements of $v\in F$ belongs to F_1, which proves that D is a φ-derivation satisfying the conditions of our theorem.

CHAPTER II. TENSOR ALGEBRAS.

Tensors are usually represented by a quantity with many indices such as $T^{ab\cdots c}_{ij\cdots k}$. However, we avoid such a representation in these lectures not only for aesthetic reasons, but also due to a more essential reason. Tensors have indices because of the use of bases; on modules without bases, such representation is impossible, while the tensor can be also defined in such cases.

To define a tensor algebra, we use the universal algebra, and then we prove the existence and uniqueness of the tensor algebra.

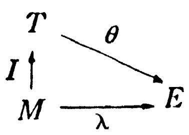
Hereafter we assume that the basic ring A is *commutative*, unless the contrary is explicitly stated.

§ 1. Tensor algebras.

DEFINITION 2.1 *Let M be a module over the basic ring A. An algebra T is called a tensor algebra over M, if it satisfies the following universality conditions:*

1) *T is an algebra containing M as a submodule, and is generated by M.*[1]

2) *For any linear mapping λ of M into an algebra E over A, there is a homomorphism θ of T into E which extends λ. This is represented in the commutative diagram:*



THEOREM 2.1. *For any module M over A, there exists always a tensor algebra T over M. T is unique under isomorphism.*

PROOF. Uniqueness: Let T, T' be two algebras with the above universality properties over M, then $T \supset M$, $T' \supset M$ and the identity mapping $I': M \to T'$ extends to a homomorphism $\theta: T \to T'$, and so

1) This means that T is generated by M and 1 in the ordinary sense. See the "Conventions".

the identity mapping $I: M \to T$ to a homomorphism $\theta': T' \to T$. The mapping $\theta' \circ \theta$ is a homomorphism $T \to T$, which coincides with the identity on M. But since M generates T, $\theta' \circ \theta$ is the identity of $T \to T$. Similarly $\theta \circ \theta'$ is the identity of $T' \to T'$, which proves that T and T' are isomorphic as algebras. Therefore the tensor algebra over M is unique under isomorphism.

Existence: First we shall construct an algebra satisfying somewhat modified condition of 2), and then we shall show that this algebra also satisfies 1).

For a while, we forget the structure of module of M and consider M as a mere *set*. In § 1, Chap. I, we have proved that there exists a free algebra F over A freely generated by the *set* M. To distinguish the addition, subtraction, multiplication and scalar multiplication in this algebra from those of M, we denote the formers by $\dot{+}$, $\dot{-}$, $\square$ and $\alpha \cdot x$ $(\alpha \in A)$ respectively. Therefore we remark that when $x, y \in M$, we have $x \dot{+} y \notin M$, $x \dot{-} y \notin M$, and $\alpha \cdot x \notin M$ in general. Next, we denote by S the set of all elements of the forms

$$x \dot{+} y \dot{-} (x+y) \qquad (x, y \in M) \tag{1}$$

and

$$\alpha \cdot x \dot{-} (\alpha x) \qquad (\alpha \in A,\ x \in M). \tag{2}$$

Let $\mathfrak{T}$ be the ideal in F generated by S. Put

$$T = F/\mathfrak{T} \qquad \text{(quotient algebra)},$$

and denote by φ the canonical mapping $F \to T$.

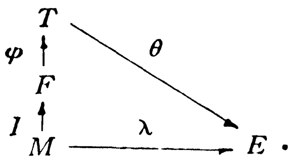
We first prove:

LEMMA 2.1. *The algebra T satisfies the following condition:*

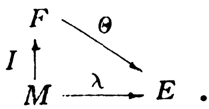
2′) If λ is a linear mapping of M into an algebra E over A, there exists a homomorphism $\theta: T \to E$ such that

$$(\theta \circ \varphi)(x) = \lambda(x) \qquad \textit{for all } x \in M. \tag{3}$$

The expression (3) *is represented in the commutative diagram where I means he injection $M \to F$:*



PROOF. By the universality of free algebras (Theorem 1.2), there exists a homomorphism $\Theta : F \to E$ which extends λ:



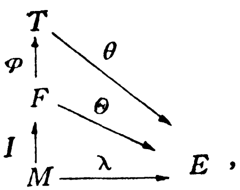
Next we prove $\Theta(\mathfrak{T})=0$. It is sufficient to prove that Θ maps all generators of $\mathfrak{T}$ upon 0. Since each generator of $\mathfrak{T}$ has the form (1) or (2), we consider them separately. In fact,

$$\begin{aligned}\Theta(x \dot{+} y \dot{-} (x+y)) &= \Theta(x)+\Theta(y)-\Theta(x+y) \\ &\qquad (\Theta \text{ is a homomorphism: } F \to E.) \\ &= \lambda(x)+\lambda(y)-\lambda(x+y) \quad (\Theta \text{ extends } \lambda.) \\ &= 0 \qquad (\lambda \text{ is linear.}),\end{aligned}$$

and similarly we have

$$\Theta(\alpha \cdot x \dot{-} \alpha x) = \alpha\,\Theta\ x) - \Theta(\alpha x) = \alpha\lambda(x) - \lambda(\alpha x) = 0,$$

which prove our assertion. Hence the kernel of Θ containing $\mathfrak{T}$, Θ defines a homomorphism $\theta : T \to E$ and if $x \in M$, we have $\theta \circ \varphi(x) = \Theta(x) = \lambda(x)$:



which proves our Lemma.

Now we shall prove that T also satisfies the condition 1) given in Definition 2.1. It is sufficient to prove that φ induces an isomorphism on M, i.e.,

(4) $$\mathfrak{T} \cap M=\{0\}\,.$$

Although (4) may be proved directly, we shall prove it using the above Lemma 2.1. Put $E=A+M$ (direct sum). Since A has a unit element 1, E is the set of elements of the form $a\cdot 1+x$, $(a\in A, x\in M)$. Define a multiplication in E by

(5) $$(a\cdot 1+x)(b\cdot 1+y)=ab\cdot 1+(bx+ay)\quad (a,b\in A\,;\; x,y\in M)\,,$$

then we have $xy=0$ for $x,y\in M$. It is easy to verify that E is an associative algebra over A with unit element, and the injection $M\to E$ is a linear univalent mapping. Therefore we have a homomorphism $\theta: T\to E$ such that

(6) $$\theta\circ\varphi(x)=x \qquad \text{for all } x\in M,$$

by Lemma 2.1. If $x\in M\cap\mathfrak{T}$, we have $\varphi(x)=0$ and then (6) asserts that $x=0$, which proves (4).

Also if $x,y\in M$; $\alpha\in A$, we have

$$\varphi(x+y)=\varphi(x\dot{+}y)=\varphi(x)+\varphi(y)\,,$$
$$\varphi(\alpha x)=\varphi(\alpha\cdot x)=\alpha\varphi(x)\,.$$

This proves that $T\supset\varphi(M)$, and then $\varphi(M)$ and M are isomorphic with each other as modules. So we identify them.[2] Then, since T is a quotient algebra of the free algebra generated by M, M and 1 form a set of generators of T. This proves that T satisfies the condition 1). Therefore the algebra T thus constructed is a tensor algebra over M, which completes our proof of existence.

EXAMPLE 1. *When M has a base consisting of only one element $\{x\}$, the tensor algebra T over $M=Ax$ is the polynomial ring $A[x]$.*

PROOF. Let T be the tensor algebra over M and P be the algebra of polynomials of X with coefficients in A. There exists a linear mapping $\lambda: M\to P$ which maps x upon X, and we have a homomorphism $\varphi: T\to P$ which extends λ. On the other hand, T being a ring generated by x and 1, an element $y\in T$ has the form

2) The identification is due to the following property: Given any set X, and a set M, there is a set Y equipotent to X which does not meet M.

$\sum a_k x^k$, and

$$\varphi\left(\sum a_k x^k\right)=\sum a_k(\varphi(x))^k=\sum a_k X^k .$$

Thus, $\varphi: T\to P$ is onto. Also, $\varphi(\sum a_k x^k)=0$ implies $\sum a_k X^k=0$, and then we must have $a_k=0$, which means that φ is an isomorphism $T\to P$. Therefore we may put $T=P=A[x]$.

§2. **Graded structure of tensor algebras.** In the above construction of the tensor algebra T over M, the ideal $\mathfrak{T}$ is generated by S whose elements are all of degree 1 in F. Hence defining all the elements of M as of degree 1, the ideal $\mathfrak{T}$ is homogeneous (cf. Theorem 1.3), and $F/\mathfrak{T}=T$ is a *graded algebra*. Decomposing F and T into homogeneous components,

$$F=\sum_h F_h, \quad \text{and} \quad T=\sum_h T_h,$$

we have

(1) $$T_h=F_h/(\mathfrak{T}\cap F_h)$$

and especially,

$$T_h=0 \text{ for } h<0, \quad T_0=A\cdot 1, \quad T_1=M.$$

Also T_h is spanned by the products of h elements of M.

We shall give a universality property of T_h as in the case of T.

THEOREM 2.2. *Let β be an h-linear mapping*[3] *of $M^h=M\times\cdots\times M$ into a module N over A. Then there exists a linear mapping ψ of T_h into N such that*

(2) $$\psi(x_1\cdots x_h)=\beta(x_1,\cdots,x_h) \text{ for all } x_1,\cdots,x_h\in M.$$

In the right hand side of (2), *$x_1\cdots x_h$ is the product of $x_1,\cdots,x_h$ in the tensor algebra T.*

PROOF. Let S be the set of generators of $\mathfrak{T}$. An element of $\mathfrak{T}$ is the sum of a finite number of elements of the form

3) An h-linear mapping means a function $\beta(x_1,\cdots,x_h)$ of h arguments $x_1,\cdots,x_h$, which is linear with respect to each argument when the other $h-1$ are kept fixed, i.e., we have

$$\beta(x_1,\cdots,x_{i-1},ax_i+bx_i',x_{i+1},\cdots,x_h)=a\beta(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_h)$$
$$+b\beta(x_1,\cdots,x_{i-1},x_i',x_{i+1},\cdots,x_h),$$
$$\text{for } a,b\in A;\ x_1,\cdots,x_h,x_i'\in M;\ i=1,\cdots,h.$$

$$a \square s \square b, \ (s \in S;\ a, b \in F),$$

where $\square$ is the free multiplication in F. Hence if $u \in F_h \cap \mathfrak{T}$, it has the form

$$u = \sum_{i=1}^{m} a_i \square s_i \square b_i, \quad (s_i \in S;\ a_i, b_i \in F),$$

and decomposing a_i and b_i into homogeneous components

$$a_i = \sum_k a_{ik}, \quad b_i = \sum_l b_{il} \quad (a_{ik} \in F_k, b_{il} \in F_l),$$

we have

$$u = \sum_{i,k,l} a_{ik} \square s_i \square b_{il}.$$

Here $a_{ik} \square s_i \square b_{il}$ is homogeneous of degree $k+l+1$, because s_i is homogeneous of degree 1. On the other hand, any homogeneous element of degree k is the sum of products of k elements of M. Therefore we have that,

(3) $u \in F_h \cap \mathfrak{T}$ is the sum of elements of the form:

$$x_1 \square \cdots \square x_k \square s \square y_1 \square \cdots \square y_l,$$

$$(k+l+1=h;\ k, l \geqq 0;\ x_1, \cdots, x_k, y_1, \cdots, y_l \in M;\ s \in S).$$

Now the $\{z_1 \square \cdots \square z_h \mid z_1, \cdots, z_h \in M\}$ forming a base of F_h, for a given h-linear mapping $\beta: M^h \to N$, there exists a linear mapping $\Psi: F_h \to N$, such that

$$\Psi(z_1 \square \cdots \square z_h) = \beta(z_1, \cdots, z_h) \text{ for all } z_1, \cdots, z_h \in M,$$

because F is free. Now we shall show that

(4) $$\Psi(\mathfrak{T} \cap F) = \{0\}.$$

In fact, by the above remark (3), it is sufficient to show that

(5) $$\Psi(x_1 \square \cdots \square x_k \square (x \dot{+} y \dot{-} (x+y)) \square y_1 \square \cdots \square y_l) = 0,$$

and

(6) $$\Psi(x_1 \square \cdots \square x_k \square (\alpha \cdot x \dot{-} \alpha x) \square y_1 \square \cdots \square y_l) = 0, \quad (k+l+1=h).$$

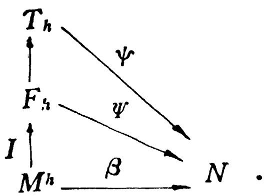
Since Ψ is linear in each of its arguments, we have

$$\Psi(x_1 \square \cdots \square x_k \square (x \dot{+} y \dot{-} (x+y)) \square y_1 \square \cdots \square y_l)$$

$$=\Psi(x_1\square\cdots\square x_k\square x\square y_1\square\cdots\square y_l)+\Psi(x_1\square\cdots\square x_k\square y\square y_1\square\cdots\square y_l)$$
$$-\Psi(x_1\square\cdots\square x_k\square(x+y)\square y_1\square\cdots\square y_l)$$
$$=\beta(x_1,\cdots,x_k,x,y_1,\cdots,y_l)+\beta(x_1,\cdots,x_k,y,y_1,\cdots,y_l)$$
$$-\beta(x_1,\cdots,x_k,x+y,y_1,\cdots,y_l)$$
$$=0 \qquad\qquad \text{(because } \beta \text{ is } h\text{-linear)},$$

and similarly we have (6), and then (4) is proved.

Thus, by (1) and (4), Ψ defines a linear mapping ψ of $T_h=F_h/(F_h\cap\mathfrak{T})$ into N, and $\Psi=\psi\circ\varphi_h=\beta$ on M^h. (φ_h is the contraction of φ to F_h). In diagrams this is represented by:



Since Ψ is not only linear, but a *homomorphism*, we have also

$$\psi(z_1\cdot z_h)=\Psi(z_1\square\cdots\square z_h)=\beta(z_1,\cdots,z_h),$$

which proves our Theorem.

Now we shall define the *tensor product* of two modules using the tensor algebra described above. A characteristic property of tensor products will be given later (cf. § 4).

DEFINITION 2.2. *Let M, N be two modules over A. We set $P=M+N$ (direct sum), and let T be the tensor algebra over P. The submodule Q of T_2 spanned by all products $\{xy\mid x\in M, y\in N\}$ is called the tensor product of M and N, and denoted by $M\otimes N$. $xy\in Q$ ($x\in M$, $y\in N$) is also denoted by $x\otimes y$.*

From Theorem 2.2, we have

COROLLARY. *Let there be given a bilinear (=2-linear) mapping β of $M\times N$ into a third module R, then there is a linear mapping ψ of Q into R, such that $\psi(x\otimes y)=\beta(x,y)$ for every $x\in M$ and $y\in N$.*

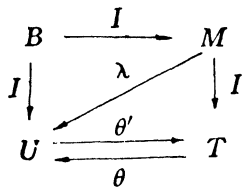
EXAMPLE 2. *If M has a base $\{x_i\}_{i\in I}=B$, then T is isomorphic to the free algebra on B. This theorem asserts that a tensor is* *represented in the form* $a_{i_1\cdots i_h}$ *if a base is determined.*

PROOF. Let U be the free algebra over B and again we use the notations $\dot{+}$, $\dot{-}$, $\square$ and $\alpha\cdot x$ for the laws of composition in U to distinguish them from the ones in M.

Let λ be linear mapping $M\to U$, which is the identity on B:

$$\lambda(a_1 x_{i_1}+\cdots+a_n x_{i_n})=a_1\cdot x_{i_1}\dot{+}\cdots\dot{+}a_n\cdot x_{i_n}.$$

Then there is a homomorphism $\theta: T\to U$ which extends λ by the property 2) of T. On the other hand, since $B\subset M\subset T$, the universality property of free algebra U asserts that there exists a homomorphism $\theta' : U\to T$ which is the identity on B. These relations are represented in the commutative diagram:



Then $\theta'\circ\theta$ is a homomorphism $T\to T$ and is the identity on B. Since B is the base of M, it is also the identity on M, and therefore also is on the algebra T generated by M. Similarly $\theta\circ\theta'$ is a homomorphism $U\to U$ and is the identity on B, and therefore also is on the algebra U generated by B. Therefore θ and θ' are isomorphisms which are reciprocal with each other. Also since λ maps M into U_1 (submodule of elements homogeneous of degree 1 in U), T is isomorphic to U not only as merely an algebra, but also as a *graded* algebra, which proves our assertion. If $\{x_i\}_{i\in I}$ is a base of M, every element in T is of the form

$$\sum_{i_1,\cdots,i_h\in I} a_{i_1\cdots i_h} x_{i_1}\cdots x_{i_h}$$

when $a_{i_1\cdots i_h}\in A$ is the component of the tensor in a familiar form.

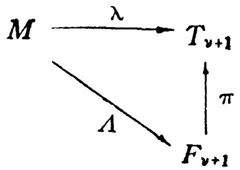
§ 3. **Derivations in a tensor algebra.** Now, we consider a module M over A and the tensor algebra T over M: $T=\sum_h T_h$. We shall prove the following

THEOREM 2.3. *If λ is a linear mapping $M \to T_{\nu+1}$ (ν : any integer $\geqq -1$), then λ may be extended uniquely to a derivation in T (of degree ν).*

PROOF. Uniqueness is obvious since M generates T. So we prove the existence of an extension. Consider the free algebra F on the set M. Then we can write $T=F/\mathfrak{I}$, $T_{\nu+1}=F_{\nu+1}/(\mathfrak{I} \cap F_{\nu+1})$, where $\mathfrak{I}$ is the ideal in F generated by the elements of the forms

$$x \dot{+} y \dot{-} (x+y) \qquad (x, y \in M),$$
$$\alpha \cdot x \dot{-} (\alpha x) \qquad (\alpha \in A, x \in M).$$

Denote by π the canonical map $F_{\nu+1} \to T_{\nu+1}$ in the factorization $T_{\nu+1}= F_{\nu+1}/(\mathfrak{I} \cap F_{\nu+1})$. For each $x \in M$, we select an element $\Lambda(x) \in F_{\nu+1}$ such that $\lambda(x)=\pi(\Lambda(x))$. This defines a map $\Lambda : M \to F_{\nu+1}$ such that the diagram



is commutative. Since M is a system of free generators of F, the map $\Lambda : M \to F_{\nu+1}$ can be extended to a derivation D of F (of degree ν). Now we shall show that

(1) $$D(\mathfrak{I}) \subset \mathfrak{I}.$$

In fact, we have

$$D(x \dot{+} y \dot{-} (x+y)) = D(x) \dot{+} D(y) \dot{-} D(x+y) \qquad (x, y \in M),$$

so that

(2) $$\pi(D(x \dot{+} y \dot{-} (x+y))) = \pi(D(x)) + \pi(D(y)) - \pi(D(x+y)).$$

But now, since $x, y, x+y$ are in M, we have

$$D(x)=\Lambda(x), \quad D(y)=\Lambda(y), \quad D(x+y)=\Lambda(x+y).$$

Therefore the right hand side of the equality (2) can be rewritten as

$$\lambda(x)+\lambda(y)-\lambda(x+y),$$

which is zero, since λ is linear. This proves that $D(x \dot{+} y \dot{-} (x+y))$

lies in the kernel of π, and therefore in $\mathfrak{T}$. Likewise we obtain $D(\alpha\cdot x\dot{-}(\alpha x))\in\mathfrak{T}$, proving (1). Thus D induces a derivation d of T in such a way that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\quad D \quad} & F \\ \pi\big\downarrow & & \big\downarrow\pi \\ T & \xrightarrow{\quad d \quad} & T \end{array}$$

(π: canonical map $F\to T$) is commutative. To see that d is an extension of λ, let $x\in M$. Then $x=\pi(x)$ and

$$d(x)=d(\pi(x))=\pi(D(x))=\pi(\Lambda(x))=\lambda(x).$$

This proves the theorem.

Tensor representation. Next, we want to make the following observation. Let M, N be modules over A, $T(M)$, $T(N)$ their tensor algebras and $\lambda: M\to N$ a linear map of M into N. Then, as a special case of the universality theorem for tensor algebras, λ extends uniquely to a homomorphism $T(M)\to T(N)$. In the special case where $M=N$, and where λ is an automorphism (i.e. an invertible linear mapping) of M, λ extends to an endomorphism $\Lambda: T(M)\to T(M)$. We assert that *this endomorphism Λ is an automorphism.* To prove this, let λ' be the inverse of λ. Then λ' extends also to an endomorphism $\Lambda': T(M)\to T(M)$, and the composite endomorphism $\Lambda\circ\Lambda': T(M)\to T(M)$ coincides with the identity on M, so that $\Lambda\circ\Lambda'=$identity on $T(M)$ which is generated by M. The same is true for $\Lambda'\circ\Lambda$. Thus Λ, with its inverse Λ', is an automorphism.

Now, the restriction of this automorphism Λ on the h-th part $T_h(M)$ of $T(M)$ gives an automorphism Λ_h of $T_h(M)$. The correspondence $\lambda\to\Lambda_h$ is a homomorphism of the group of automorphisms of M into that of the module $T_h(M)$. This homomorphism we call the *tensor representation* of degree h.

REMARK. Suppose M is a submodule of N, for which the injection map $M\to N$ is denoted by λ. Then the homomorphism $\Lambda: T(M)\to T(N)$ induced by λ is, in general, not an isomorphism. However, in some special cases, Λ is an isomorphism; for example, in case where N is the direct sum of M and some other module $P: N=M+P$ (direct), or in case where both M, N have free bases.

The following provides an example of which Λ is not an isomorphism. Let $A=Z$ be the ring of integers, $N=\{0, 1, 2, 3\}$ the cyclic group of order 4, and let $M=\{0, 2\}$ be the subgroup of N of index 2. Then Λ maps the non-zero element $2 \otimes 2$ of $M \otimes M=M$ upon the zero element of $N \otimes N=N$, for we have $\Lambda(2 \otimes 2)=2 \otimes 2=4(1 \otimes 1)=0$. This shows that $\Lambda: T(M) \rightarrow T(N)$ is not an isomorphism.

§ 4. **Preliminaries on tensor product of modules.** Before considering tensor product of semi-graded algebras, we give here some preliminaries on tensor product of modules.

Characterization. Let $M_1, \cdots, M_h$ be modules over A. Then the tensor product $P=M_1 \otimes \cdots \otimes M_h$ can be characterized in the following manner:

1) *P is a module over A into which there is an h-linear map*

$$\alpha: M_1 \times \cdots \times M_h \rightarrow P$$

such that the elements $\alpha(x_1, \cdots, x_h)=x_1 \otimes \cdots \otimes x_h \in P$ $(x_i \in M_i, i=1, \quad, h)$ span P.

Here we say that the map α is h-linear if $\alpha(x_1, \cdot \;, x_h)=x_1 \otimes \cdots \otimes x_h \in P\,(x_i \in M_i,\; i=1, \cdots, h)$ depends linearly on each one of the entries $x_1, \cdot\cdot, x_h$ when the others are fixed.

2) *If β is an h-linear mapping of $M_1 \times \cdots \times M_h$ into a module Q, then there is a linear map $\varphi: P \rightarrow Q$ such that $\varphi \circ \alpha=\beta$.*

Associativity and commutativity. Let $M_1, \cdot\cdot, M_k, M_{k+1}, \cdots, M_h$ $(1 \leqq k < h)$ be modules over A, and put $P=M_1 \otimes \cdots \otimes M_h$, $P'=(M_1 \otimes \quad \otimes M_k) \otimes (M_{k+1} \otimes \quad \otimes M_h)$. Then *there is an isomorphism $P \rightarrow P'$ which maps $x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdot\cdot \otimes x_h$ upon $(x_1 \otimes \cdot\cdot \otimes x_k) \otimes (x_{k+1} \otimes \cdot \otimes x_h)$ for any $x_i \in M_i$ $(i=1, \cdot\cdot, h)$.*

Since we have given the characteristic properties 1), 2) for the tensor product, we need only to prove 1) that $(x_1 \otimes \cdot \otimes x_k) \otimes (x_{k+1} \otimes \cdots \otimes x_h) \in P'$ $(x_i \in M_i, i=1, \cdots, h)$ depends linearly on each argument, and P' is spanned by elements of the above form, and 2) that, if

β is a multilinear map $M_1 \times \cdots \times M_h \to Q$, then there is a linear map $\varphi : P' \to Q$ such that

$$\varphi((x_1 \otimes \cdots \otimes x_k) \otimes (x_{k+1} \otimes \cdots \otimes x_h)) = \beta(x_1, \cdots, x_h).$$

1) is obvious. In order to construct the map $\varphi : P' \to Q$, we consider first the mapping

$$(x_1, \cdots, x_k) \to \beta(x_1, \cdots, x_k, x_{k+1}, \cdots, x_h)$$

for each set of fixed values of $x_{k+1}, \cdots, x_h$. This mapping is a k-linear map: $M_1 \times \cdots \times M_k \to Q$. Therefore, there is a linear map, say $\psi_{x_{k+1}, \cdots, x_h} :\ M_1 \otimes \cdots \otimes M_k \to Q$, such that

$$\psi_{x_{k+1}, \ldots, x_h}(x_1 \otimes \cdots \otimes x_k) = \beta(x_1, \cdots, x_k, x_{k+1}, \cdots, x_h).$$

Now, let t be any element in $M_1 \otimes \cdots \otimes M_k$. For this fixed t, we consider the mapping

$$(x_{k+1}, \cdots, x_h) \to \psi_{x_{k+1}, \cdots, x_h}(t).$$

We assert that this is a multilinear mapping. In fact, this is true if t is of the form $t = x_1 \otimes \cdots \otimes x_k$, because in that case we have

$$\psi_{x_{k+1}, \cdots, x_h}(t) = \beta(x_1, \cdots, x_k, x_{k+1}, \cdots, x_h).$$

Let now $t = \sum \alpha_i t_i$, where each t_i is of the form $x_1 \otimes \cdots \otimes x_k$. Since $\psi_{x_{k+1}, \cdots, x_h} :\ M_1 \otimes \cdots \otimes M_k \to Q$ is linear, we obtain

$$\psi_{x_{k+1}, \cdots, x_h}(t) = \sum_i \alpha_i \psi_{x_{k+1}, \ldots, x_h}(t_i).$$

Each summand $\alpha_i \psi_{x_{k+1}, \cdots, x_h}(t_i)$ being multilinear in $(x_{k+1}, \cdots, x_h)$, we can conclude that $\psi_{x_{k+1}, \cdots, x_h}(t)$ is multilinear in $(x_{k+1}, \cdots, x_h)$. Thus, for given $t \in M_1 \otimes \cdots \otimes M_k$, there is a linear map $\gamma_t :\ M_{k+1} \otimes \cdots \otimes M_h \to Q$ such that $\gamma_t(x_{k+1} \otimes \cdots \otimes x_h) = \psi_{x_{k+1}, \cdots, x_h}(t)$.

Similarly, we can prove that, for any fixed element $u \in M_{k+1} \otimes \cdots \otimes M_h$, the mapping $t \to \gamma_t(u)$ is linear. Thus, the mapping $(t, u) \to \gamma_t(u)$ is a bilinear map: $(M_1 \otimes \cdots \otimes M_k) \times (M_{k+1} \otimes \cdots \otimes M_h) \to Q$, and so, there is a linear map $\varphi : (M_1 \otimes \cdots \otimes M_k) \otimes (M_{k+1} \otimes \cdots \otimes M_h) \to Q$, such that

$$\varphi(t \otimes u) = \gamma_t(u) \quad (t \in M_1 \otimes \cdots \otimes M_k,\ u \in M_{k+1} \otimes \cdots \otimes M_h)$$

Thus, for $t = x_1 \otimes \cdots \otimes x_k$, $u = x_{k+1} \otimes \cdots \otimes x_h$, we have

$$\varphi((x_1 \otimes \cdots \otimes x_k) \otimes (x_{k+1} \otimes \cdots \otimes x_h))$$
$$= \beta(x_1, \cdots, x_k, x_{k+1}, \cdots, x_h),$$

which proves 2). Thus our assertion is proved.

By identifying $x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_h$ with

$(x_1 \otimes \cdots \otimes x_k) \otimes (x_{k+1} \otimes \cdots \otimes x_h)$, we take

$$M_1 \otimes \cdots \otimes M_h = (M_1 \otimes \cdots \otimes M_k) \otimes (M_{k+1} \otimes \cdots \otimes M_h).$$

Let again $M_1, \cdots, M_h$ be modules over A, and let π be any permutation of $\{1, \cdots, h\}$. Then there is an isomorphism λ_π of $M_1 \otimes \cdots \otimes M_h$ onto $M_{\pi(1)} \otimes \cdots \otimes M_{\pi(h)}$ such that

$$\lambda_\pi(x_1 \otimes \cdots \otimes x_h) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(h)} \quad (x_i \in M_i, i=1, \cdots, h).$$

In fact, since the mapping

$$(x_1, \cdots, x_h) \to x_{\pi(1)} \otimes \cdots \otimes x_{\pi(h)}$$

is multilinear, there exists a linear map $\lambda_\pi : M_1 \otimes \cdots \otimes M_h \to M_{\pi(1)} \otimes \cdots \otimes M_{\pi(h)}$ such that

$$\lambda_\pi(x_1 \otimes \cdots \otimes x_h) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(h)}.$$

So it remains only to prove that λ_π is invertible. Let $\lambda'_\pi : M_{\pi(1)} \otimes \cdots \otimes M_{\pi(h)} \to M_1 \otimes \cdots \otimes M_h$ be the linear map obtained similarly from the multilinear mapping

$$(x_{\pi(1)}, \cdots, x_{\pi(h)}) \to x_1 \otimes \cdots \otimes x_h.$$

Then

$$\lambda'_\pi(x_{\pi(1)} \otimes \cdots \otimes x_{\pi(h)}) = x_1 \otimes \cdots \otimes x_h,$$

so that

$$\lambda_\pi \circ \lambda'_\pi = \text{identity of } M_{\pi(1)} \otimes \cdots \otimes M_{\pi(h)},$$
$$\lambda'_\pi \circ \lambda_\pi = \text{identity of } M_1 \otimes \cdots \otimes M_h.$$

This proves that λ_π, with its inverse λ'_π, is an isomorphism onto.

REMARK. Identification of $(x_1 \otimes \cdots \otimes x_h) \otimes (x_{h+1} \otimes \cdots \otimes x)$ with $x_1 \otimes \cdots \otimes x_h$ in the case of associativity does not cause any confusion, while, identification will not be permitted in the case of commutativity. The reader must be careful not to make the following sort of mistakes. Consider the case $M_1 = M_2 = M$, $M \ni x_1, x_2$. Can we identify $x_2 \otimes x_1$ with $x_1 \otimes x_2$ in $M \otimes M$? No! These two elements are by no means identical in general.

§5. **Tensor product of semi-graded algebras.** Let E, E' be semi-graded algebras over A:

$$E=E_+ + E_-, \quad E'=E'_+ + E'_- .$$

Now, we shall give $E\otimes E'$, the tensor product of the modules E, E', a structure of semi-graded algebra. To do this, we first define the multiplication in $E\otimes E'$, in terms of a bilinear map $(E\otimes E')\times(E\otimes E')\to E\otimes E'$.

Since $(E\otimes E'_+)+(E\otimes E'_-)=E\otimes E'=(E_+\otimes E')+(E_-\otimes E')$, it suffices to define four bilinear maps:

$$(E\otimes E'_+)\times(E_+\otimes E') \to E\otimes E',$$
$$(E\otimes E'_+)\times(E_-\otimes E') \to E\otimes E',$$
$$(E\otimes E'_-)\times(E_+\otimes E') \to E\otimes E',$$
$$(E\otimes E'_-)\times(E_-\otimes E') \to E\otimes E',$$

which will be well defined as soon as quadri-linear maps:

$$E\times E'_+\times E_+\times E' \to E\otimes E',$$
$$E\times E'_+\times E_-\times E' \to E\otimes E',$$
$$E\times E'_-\times E_+\times E' \to E\otimes E',$$
$$E\times E'_-\times E_-\times E' \to E\otimes E',$$

are given. The first three maps are defined by

$$(x,x',y,y')\mapsto xy\otimes x'y' \quad \begin{cases} x\in E,\ y'\in E', \text{ and either} \\ \quad x'\in E'_+,\ y\in E_+ \\ \text{or}\quad x'\in E'_+,\ y\in E_- \\ \text{or}\quad x'\in E'_-,\ y\in E_+, \end{cases}$$

while, the last one is defined by

$$(x,x',y,y')\to-(xy)\otimes(x'y') \quad (x\in E,\ x'\in E'_-,\ y\in E_-,\ y'\in E').$$

In this way, we obtain a bilinear multiplication $(E\otimes E')\cdot(E\otimes E')\subset E\otimes E'$. Now we assert that this *multiplication is associative.* Since every element of $E\otimes E'$ is a linear combination of elements of the form $x\otimes x'$, where both x and x' are homogeneous in the semi-gradations, it will be sufficient to check the associativity of the multiplication for elements of that form. For convenience' sake, we set

$$\varepsilon(x)=\begin{cases}0 & \text{if} \quad x\in E_+ ,\\ 1 & \text{if} \quad x\in E_- ,\end{cases}$$

where 0, 1 denote the elements of the gradation group $\Gamma=\{0, 1\}$. Then we have

$$\varepsilon(xy)=\varepsilon(x)+\varepsilon(y), \text{ if both } x, y \text{ are homogeneous.}$$

Similarly we define $\varepsilon'(x')$ for any homogeneous element $x'\in E'$. Then, as is easily seen, we have

$$(1) \qquad (x\otimes x')\cdot(y\otimes y')=(-1)^{\varepsilon'(x')\varepsilon(y)}(xy)\otimes(x'y')^{4)}$$

$$(x\in E, y'\in E', \ x' \text{ homogeneous } \in E', \ y \text{ homogeneous } \in E).$$

Now we check the identity

$$(2) \qquad ((x\otimes x')\cdot(y\otimes y'))\cdot(z\otimes z')=(x\otimes x')\cdot((y\otimes y')\cdot(z\otimes z'))$$

$$(x, y, z \text{ homogeneous } \in E, \text{ and } x', y', z' \text{ homogeneous } \in E').$$

Computing the left hand side of (2), we obtain

$$\begin{aligned}((x\otimes x')\cdot(y\otimes y'))\cdot(z\otimes z')&=(-1)^{\varepsilon'(x')\varepsilon(y)}(xy\otimes x'y')\cdot(z\otimes z')\\ &=(-1)^{\varepsilon'(x')\varepsilon(y)+\varepsilon'(x'y')\varepsilon(z)}(xyz\otimes x'y'z')\\ &=(-1)^{\varepsilon'(x')\varepsilon(y)+\varepsilon'(x')\varepsilon(z)+\varepsilon'(y')\varepsilon(z)}(xyz\otimes x'y'z'),\end{aligned}$$

while, the right hand side of (2) can be reduced as follows:

$$\begin{aligned}(x\otimes x')\cdot((y\otimes y')\cdot(z\otimes z'))&=(-1)^{\varepsilon'(y')\varepsilon(z)}(x\otimes x')(yz\otimes y'z')\\ &=(-1)^{\varepsilon'(y')\varepsilon(z)+\varepsilon'(x')\varepsilon(yz)}(xyz\otimes x'y'z')\\ &=(-1)^{\varepsilon'(y')\varepsilon(z)+\varepsilon'(x')\varepsilon(y)+\varepsilon'(x')\varepsilon(z)}(xyz\otimes x'y'z').\end{aligned}$$

This proves the associativity of the multiplication. If 1, $1'$ are the multiplicative units in E, E' respectively, then it is clear that $1\otimes 1'\in E\otimes E'$ is the multiplicative unit in $E\otimes E'$.

Thus $E\otimes E'$ is an associative algebra, which is semi-graded, namely, if we put

$$(E\otimes E')_+=(E_+\otimes E'_+)+(E_-\otimes E'_-),$$

---

4) See p. 9, (the definition of $(-1)^{\nu\nu'}$).

$$(E \otimes E')_- = (E_+ \otimes E'_-) + (E_- \otimes E'_+),$$

then

$$E \otimes E' = (E \otimes E')_+ + (E \otimes E')_-,$$

and

$$(E \otimes E')_+ \cdot (E \otimes E')_+ \subset (E \otimes E')_+,$$

$$(E \otimes E')_+ \cdot (E \otimes E')_- \subset (E \otimes E')_-,$$

$$(E \otimes E')_- \cdot (E \otimes E')_+ \subset (E \otimes E')_-,$$

$$(E \otimes E')_- \cdot (E \otimes E')_- \subset (E \otimes E')_+.$$

Observe that, if E, E' are Γ-graded algebras of which a fixed subgroup Γ_+ of Γ of index 2 is given, then by the associated semi-gradation

$$E_+ = \sum_{\gamma \in \Gamma_+} E_\gamma, \quad E_- = \sum_{\gamma \in \Gamma_-} E_\gamma,$$

$$E'_+ = \sum_{\gamma \in \Gamma_+} E'_\gamma, \quad E'_- = \sum_{\gamma \in \Gamma_-} E'_\gamma,$$

$E \otimes E'$ is a semi-graded algebra. The associative algebra $E \otimes E'$ also admits the following Γ-gradation:

$$E \otimes E' = \sum_{\beta \in \Gamma} (E \otimes E')_\beta, \text{ where}$$

$$(E \otimes E')_\beta = \sum_{\gamma + \gamma' = \beta} E_\gamma \otimes E'_{\gamma'},$$

of which the associated semi-gradation is just the semi-gradation of $E \otimes E'$ given above. Direct definition of the multiplication in the Γ-graded algebra $E \otimes E'$ is given by

$$(x \otimes x') \cdot (y \otimes y') = (-1)^{\gamma' \beta} xy \otimes x'y' \quad (x \in E, x' \in E'_{\gamma'}, y \in E_\beta, y' \in E').$$

CHAPTER III. CLIFFORD ALGEBRAS.

§ 1. **Clifford algebras.** A Clifford algebra is an algebra associated to a quadratic form $f(x)$, and, roughly speaking, the one satisfying

$$x^2 = f(x) \cdot 1. \tag{1}$$

First we define a quadratic form without using the base of module.

DEFINITION 3.1. *Let M be a module over the basic ring A. A quadratic form on M is a mapping $f: M \to A$ such that*

1) $f(\alpha x) = \alpha^2 f(x)$ for all $\alpha \in A$, $x \in M$;

2) the mapping $(x, y) \to f(x+y) - f(x) - f(y) = \beta(x, y)$ of $M \times M$ into A is bilinear. $\beta(x, y)$ is called the bilinear form associated to f.

It is evident from the definition, that β is symmetric:

$$\beta(x, y) = \beta(y, x) \quad \text{and} \quad \beta(x, x) = 2\, f(x).$$

Two elements x, y such that $\beta(x, y) = 0$ is said to be *orthogonal* with each other. When M is an n-dimensional vector space over A with a base $(x_1, \cdots, x_n)$ and if $f(x) = f(\sum_{i=1}^{n} \xi_i x_i) = \xi_1^2 + \cdots + \xi_n^2$, then we have

$$\beta(x, y) = \beta(\sum \xi_i x_i, \sum \eta_i x_i) = 2\, (\xi_1 \eta_1 + \cdots + \xi_n \eta_n).$$

Hence the above definition of orthogonality coincides with the ordinary one in the n-dimensional space.

Hereafter we assume that there is given a quadratic form $f(x)$.

DEFINITION 3.2. *Let T be the tensor algebra over M, and denote by $\otimes$ the multiplication*[1] *in T. Let $\mathfrak{C}$ be the ideal generated in T by all the elements of the form*

$$x \otimes x - f(x) \cdot 1, \qquad x \in M, \tag{2}$$

1) In this chapter, we use this notation to distinguish it from the other various multiplications which will be considered later.

where 1 *is the unit of* T. *The quotient algebra* $C=T/\mathfrak{C}$ *is called the Clifford algebra associated to* f *over* M.

If π is the canonical mapping $T\to C$, $\pi(M)$ is a submodule of C, which generates C. Also we have

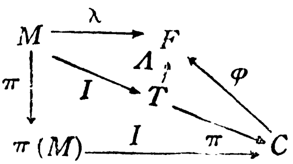
$$(\pi(x))^2=f(x)\cdot 1 \quad \text{if} \quad x\in M.$$

We remark that the kernel of π in M is not always 0, and we *cannot identify* M and $\pi(M)$ in general. However, if we wish to construct an algebra satisfying (1), the universality leads to this definition as is shown in the following:

THEOREM 3.1. *Assume that we have a linear mapping* λ *of* M *into an algebra* F *such that* $(\lambda(x))^2=f(x)\cdot 1$ *for all* $x\in M$. *Then there exists a homomorphism* φ *of* C *into* F *such that*

$$\lambda(x)=\varphi(\pi(x)), \quad \textit{for all} \quad x\in M.$$

This is represented in the diagram:



PROOF. The definition of the tensor algebra asserts the existence of a mapping $\Lambda: T\to F$ which extends λ. If $x\in M$, we have

$$\Lambda(x\otimes x-f(x)\cdot 1)=(\lambda(x))^2-f(x)\cdot 1=0.$$

Thus the generator of $\mathfrak{C}$ being mapped upon 0, we have $\Lambda(\mathfrak{C})=0$, which proves that Λ defines a homomorphism φ of C into F satisfying $\Lambda=\varphi\circ\pi$. The contraction λ of Λ into M satisfies our requirements.

If we put $f(x)=g(\pi(x))$, g is a quadratic form in $\pi(M)$ into A, and for $y\in\pi(M)$, we have

$$y^2=g(y)\cdot 1.$$

Semi-graded structure of Clifford algebras. We have described in the previous chapter, that the tensor algebra T is graded, and *a fortiori*, is a semi-graded algebra. Since the element $x\otimes x$ or $f(x)\cdot 1$

is of degree 2 or 0 respectively, the elements (2) is homogeneous in the *semi-gradation* of T. Decomposing T into $T_+ + T_-$, (2) belongs to T_+, and $\mathfrak{C}$ is homogeneous in the semi-gradation of T, which proves that $C = T/\mathfrak{C}$ is a *semi-graded algebra*. Putting $C = C_+ + C_-$, C_+ and C_- are generated by the products of even and odd numbers of elements of $\pi(M)$ respectively, because

$$C_+ = \sum_{h:\text{even}} \pi(T_h) \quad \text{and} \quad C_- = \sum_{h:\text{odd}} \pi(T_h).$$

If we put $\bar{x} = \omega'x)$ for $x \in M$, we have $\bar{x}^2 = f(x) \cdot 1$, and then

$$\begin{aligned} \bar{x}\bar{y} + \bar{y}\bar{x} &= (\bar{x}+\bar{y})^2 - \bar{x}^2 - \bar{y}^2 \\ &= f(x+y) \cdot 1 - f(x) \cdot 1 - f(y) \cdot 1 = \beta(x, y) \cdot 1. \end{aligned} \tag{3}$$

Therefore, if x and y are orthogonal, we obtain

$$\bar{x}\bar{y} + \bar{y}\bar{x} = 0, \quad \text{or} \quad \bar{x}\bar{y} = -\bar{y}\bar{x}. \tag{4}$$

§ 2. Exterior algebras.

DEFINITION 3.3. *When the quadratic form f reduces to* 0, *the Clifford algebra C associated to $f=0$, is called the exterior algebra over M.*

We have easily

$$xx = 0 \tag{1}$$

and

$$xy + yx = 0, \quad \text{or} \quad xy = -yx, \tag{2}$$

in the case of exterior algebra. The generators of $\mathfrak{C}$ reduces to $x \otimes x \in T_2$ which are homogeneous not only in the semi-gradation of T, but also in the *graded* structure of T, so that the exterior algebra $E = T/\mathfrak{C}$ has the structure of a *graded* algebra.

THEOREM 3.2. *In the case of exterior algebra E, π (the canonical mapping of T into E) is an isomorphism on M, and identifying M with $\pi(M)$, we may imbed M into E.*

PROOF. Since the elements of $\mathfrak{C}$ are linear combinations of sums of elements of the form

$$u \otimes (x \otimes x) \otimes v$$

where $x \in M$, and u, v are homogeneous in T. If $u \in T_h$, $v \in T_k$, we have $u \otimes (x \otimes x) \otimes v \in T_{h+k+2}$ and then this has a degree not lses than 2 provided that $u \neq 0$ and $v \neq 0$. Therefore the homogeneous components of an element of $\mathfrak{C}$ which are not 0 must be of degree $\geqq 2$. On the other hand, the elements of M being of degree 1, we have $\mathfrak{C} \cap M=\{0\}$, which proves that π is an isomorphism of M.

Hence we identify M with its image under π in E. Then we have $E_0=A\cdot 1$, $E_1=M$. For $h>1$, E_h is spanned by the products of h elements of M, i.e., by the elements $x_1\cdots x_h$, where $x_i \in M$.

§ 3. **Structure of the Clifford algebras when M has a base.** Let M be a module over A and f a quadratic form on M. Let $C=T/\mathfrak{C}$ be the Clifford algebra associated to f over M.

1°. First we consider the case $M=A\cdot x$ (i.e., M is generated by a single element x). As we have already proved in § 1, Chap. II, the tensor algebra T over $M=A\cdot x$ is the polynomial ring$A$ $[x]$, and $\mathfrak{C}$ is generated by $x^2-f(x)\cdot 1$. If we denote by ξ the class of x under π, $C=T/\mathfrak{C}$ has the form $A+A\cdot\xi$ where $\xi^2=f(\xi)\cdot 1$. Hence $A\cdot\xi$ being a free module with a base ξ, the canonical mapping $M\to C$ is an isomorphism $A\cdot x\to A\cdot\xi \subset C$. Therefore we may imbed M into C in this case.

2°. Next we consider the case where $M=N+P$ (direct sum), and N and P are orthogonal with each other; i.e.,

$$\beta(x, y)=0 \text{ for all } x\in N,\ y\in P.$$

By the orthogonality property, we have

$$f(x+y)=f(x)+f(y) \text{ if } x\in N \text{ and } y\in P. \tag{1}$$

THEOREM 3.3. *Under such conditions, let C_M, C_N and C_P be the Clifford algebras over M, N and P associated to f or the restrictions of f on N and P respectively. Then we have*

$$C_M=C_N \otimes C_P \qquad (\textit{tensor product}). \tag{2}$$

PROOF. Let T_M, T_N and T_P be the tensor algebras over M, N and P and π_M, π_N, π_P the canonical mappings of $T_M\to C_M$, $T_N\to C_N$, $T_P\to C_P$ respectively. By the definition of tensor algebra, the injection mapping $\varphi: N\to M$ is extended to a homomorphism $\tilde{\varphi}: T_N\to T_M$, and since

$$\tilde{\varphi}(x \underset{N}{\otimes} x - f(x)\cdot 1) = x \underset{M}{\otimes} x - f(x)\cdot 1, \text{ for } x \in N,$$

$\tilde{\varphi}$ defines a homomorphism of C_N into C_M, which will be denoted also by φ. Similarly we have a homomorphism ψ of C_P into C_M, which extends the injection mapping $P \to M$.

$$\begin{array}{ccccc} N & \xrightarrow{I} & T_N & \xrightarrow{\pi_N} & C_N \\ \varphi \downarrow & & \tilde{\varphi} \downarrow & & \downarrow \varphi \\ M & \xrightarrow[I]{} & T_M & \xrightarrow[\pi_M]{} & C_M \end{array}$$

The product $\varphi(u)\psi(v)$ in C_M being bilinear with respect to $u \in C_N$, $v \in C_P$, we have, by the characteristic property of tensor product, a linear mapping θ of the module $C_N \otimes C_P$ into C_M such that

$$(3) \qquad \theta(u \otimes v) = \varphi(u)\psi(v) \qquad (u \in C_N,\ v \in C_P).$$

By the orthogonality of N and P, we have for $x \in N$, $y \in P$,

$$(4) \qquad \bar{x}\bar{y} = -\bar{y}\bar{x}$$

where $\bar{x} = \pi_M(\varphi(x)) = \varphi(\pi_N(x))$ and $\bar{y} = \pi_M(\psi(y)) = \psi(\pi_P(y))$.

Now $C_N = (C_N)_+ + (C_N)_-$ (semi-graded), where $(C_N)_+$, $(C_N)_-$ are spanned by the products of even or odd numbers of elements of $\pi_N(N)$ respectively. Similarly we put $C_P = (C_P)_+ + (C_P)_-$. By the anti-commutativity (4), we have

$$(5) \qquad \begin{cases} \varphi(u)\psi(v) = \psi(v)\varphi(u) & \text{if either } u \in (C_N)_+ \text{ or } v \in (C_P)_+, \\ \varphi(u)\psi(v) = -\psi(v)\varphi(u) & \text{if both } u \in (C_N)_- \text{ and } v \in (C_P)_-. \end{cases}$$

Here we shall show that

LEMMA 3.1. *The linear mapping θ defined above is a homomorphism of $C_N \otimes C_P \to C_M$, i.e., θ satisfies*

$$(6) \qquad \theta((u \otimes v)(u' \otimes v')) = \theta(u \otimes v)\ \theta(u' \otimes v'),$$
$$u, u' \in C_N;\ v, v' \in C_P,$$

where the term in the parentheses in the left hand side of (6) *is the product of $u \otimes v$ and $u' \otimes v'$ in $C_N \otimes C_P$ which has been defined in* § 5, *Chap. II.*

PROOF. It is sufficient to prove that (6) holds when u, v, u', v' are all homogeneous in the semi-graded structure.

Putting

$$\eta=\begin{cases}0 & \text{if } v\in(C_P)_+\\ 1 & \text{if } v\in(C_P)_-,\end{cases}$$

and

$$\varepsilon'=\begin{cases}0 & \text{if } u'\in(C_N)_+\\ 1 & \text{if } u'\in(C_N)_-,\end{cases}$$

we have $(u\otimes v)(u'\otimes v')=(-1)^{\eta\varepsilon'}uu'\otimes vv'$ by the definition of the product in the tensor algebra (§ 5, Chap. II). Then we have

$$\begin{aligned}\theta((u\otimes v)(u'\otimes v'))&=(-1)^{\eta\varepsilon'}\theta(uu'\otimes vv')\\ &=(-1)^{\eta\varepsilon'}\varphi(uu')\psi(vv') \quad (\text{by } (3))\\ &=(-1)^{\eta\varepsilon'}\varphi(u)\varphi(u')\psi(v)\psi(v')\end{aligned}$$

(since φ and ψ are homomorphism).

On the other hand (5) is equivalent to

$$(5')\qquad \psi(v)\varphi(u')=(-1)^{\eta\varepsilon'}\varphi(u')\psi(v),$$

and then

$$\begin{aligned}\theta(u\otimes v)\theta'(u'\otimes v')&=\varphi(u)\psi(v)\varphi(u')\psi(v') \quad (\text{by } (3))\\ &=(-1)^{\eta\varepsilon'}\varphi(u)\varphi(u')\psi(v)\psi(v') \quad (\text{by } (5'))\end{aligned}$$

which proves our assertion (6).

After having constructed a homomorphism $\theta: C_N\otimes C_P\rightarrow C_M$, we next construct a homomorphism of the inverse direction $\lambda: C_M\rightarrow C_N\otimes C_P$. First define a linear mapping $\lambda_0: M=N+P\rightarrow C_N\otimes C_P$ by

$$(7)\qquad \lambda_0(x+y)=\pi_N(x)\otimes 1+1\otimes\pi_P(y)\quad (x\in N,\ y\in P),$$

where 1 is the unit in C_P or C_N. $C_N\otimes C_P$ being an algebra, we have

$$(\lambda_0(x+y))^2=(\pi_N(x)\otimes 1)^2+(1\otimes\pi_P(y))^2+\pi_N(x)\otimes\pi_P(y)$$
$$+\pi_P(y)\otimes\pi_N(x)$$

and since $\pi_N(x)\in(C_N)_-$, $\pi_P(y)\in(C_P)_-$, the last two terms cancel out with each other by (4). Also

$$(\pi_N(x)\otimes 1)^2=(\pi_N(x))^2\otimes 1=f(x)(1\otimes 1),$$

and similarly $(1\otimes\pi_P(y))^2=f(y)(1\otimes 1)$, then we have

$$(\lambda_0(x+y))=f(x)(1\otimes 1)+f(y)(1\otimes 1)$$
$$=f(x+y)(1\otimes 1) \qquad (\text{by } (1)),$$

i. e., we obtain

(8) $$(\lambda_0(z))^2-f(z)(1\otimes 1)=0\,,\ (z\in M)\,.$$

λ_0 is extended to a homomorphism $\tilde{\lambda}: T_M\to C_N\otimes C_P$ and classifying by $\mathfrak{C}_M$, λ_0 defines at last a homomorphism $\lambda: C_M\to C_N\otimes C_P$ satisfying

(9) $$\lambda(\pi_M(z))=\lambda_0(z) \qquad \text{for all } z\in M,$$

because of (8).

We remark that

(10) $$\theta(\pi_N(x)\otimes 1)=\varphi(\pi_N(x))\psi(1)=\pi_M(\varphi(x))\cdot 1=\pi_M(x)$$

by (3). Now we have by (9), (7) and (10),

$$\lambda\circ\theta(\pi_N(x)\otimes 1)=\lambda(\pi_M(x))=\lambda_0(x)=\pi_N(x)\otimes 1\,,$$

and similarly $\lambda\circ\theta(1\otimes\pi_P(y))=1\otimes\pi_P(y)$. But since $C_N\otimes C_P$ is generated by $\pi_N(x)\otimes 1$ and $1\otimes\pi_P(y)$, the homomorphism $\lambda\circ\theta$ is the identity on $C_N\otimes C_P$. On the other hand, we have by (9), (7), and (10)

$$\theta\circ\lambda(\pi_M(x+y))=\theta(\lambda_0(x+y))=\theta(\pi_N(x)\otimes 1)+\theta(1\otimes\pi_P(y))$$
$$=\pi_M(x)+\pi_M(y)=\pi_M(x+y) \quad (x\in N,\ y\in P)\,,$$

and since $\pi_M(x+y)$'s generate C_M, the homomorphism $\theta\circ\lambda$ is also the identity on C_M. Hence C_M and $C_N\otimes C_P$ are isomorphic with each other, which proves our theorem.

3°. When A is a field K of characteristic $\neq 2$, and M is of dimension 2 over K, it is well known that f is represented in a form

$$f(\xi x+\eta y)=a\xi^2+b\eta^2 \qquad (a, b\in K)\,,$$

by a suitable choice of a base x, y. If we put $N=K\cdot x$, $P=K\cdot y$, x and y are orthogonal, since f does not contains the term $\xi\eta$. Therefore we have $C_M=C_N\otimes C_P$, and since N or P is generated by only one element x or y respectively, the consideration in 1° gives now

$$C_N=K+Kx\,, \qquad C_P=K+Kv$$

Thus we obtain

$$C_M=(K+Kx)\otimes(K+Ky)=K+K\otimes Ky+Kx\otimes K+Kx\otimes Ky\,.$$

which proves that C_M is spanned by four elements $1\otimes 1=1$, $1\otimes y$, $x\otimes 1$, and $x\otimes y$. The products between these basic elements are given by the following:

$$(x\otimes 1)^2=x^2\otimes 1=f(x)\cdot 1=a\cdot 1\,,$$
$$(1\otimes y)^2=1\otimes y^2=f(y)\cdot 1=b\cdot 1\,,$$
$$(x\otimes 1)(1\otimes y)=x\otimes y=-(1\otimes y)(x\otimes 1)\,,$$

(since both $x\otimes 1$ and $1\otimes y$ are of degree 1).

Putting $x\otimes 1=X$, $1\otimes y=Y$, we have $x\otimes y=XY$, and the products are given by

$$X^2=a\,,\quad Y^2=b\,,\quad XY=-YX\,.$$

This is nothing but a generalized quaternion algebra over K. In the case where $a=b=-1$ and K is the real number field, this is the ordinary quaternion algebra of Hamilton.

4°. Suppose that M has a base consisting of a finite number of elements $(x_1,\cdots,x_n)$ which are mutually orthogonal:

$$\beta(x_i, x_j)=0\,,\ (i\neq j)\,.$$

It is well known in the theory of quadratic forms, that when A is a field of characteristic $\neq 2$, we can always take such a base.[2)]

THEOREM 3.4. *Under such assumptions, M is identified with the submodule $\pi(M)$ of the Clifford algebra C_M over M. Also C_M is spanned by the elements $x_{i_1}\cdots x_{i_h}(i_1<\cdots<i_h)$.*

PROOF. Since this is proved when $n=1$ in 1°, we proceed by induction of n, and assume that this statement has already been proved for $n-1$. Put $N=Ax_1+\cdots+Ax_{n-1}$, and $P=Ax_n$; N and P satisfy the assumptions of Theorem 3.3, so we have $C_M\cong C_N\otimes C_P$. Under this isomorphism, $\pi_M(x+y)$ corresponds to $\pi_N(x)\otimes 1+1\otimes \pi_P(y)$, $(x\in N,\ y\in P)$. The assumption of the induction asserts the identification of x with $\pi_N(x)$ and y with $\pi_P(y)$. Also $x\otimes 1+1\otimes y$

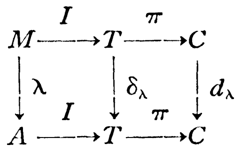
2) In the case of characteristic 2, such a base exists only in the trivial case where the quadratic form $f(x)$ is the square of a linear form.

being 0 if and only if $x=y=0$, the correspondence $M \ni (x+y) \rightarrow x \otimes 1 + 1 \otimes y = \pi_M(x+y)$ is an isomorphism. Thus M may be identified with $\pi_M(M)$. Next by our assumption of induction, C_N is spanned by the elements $x_{j_1} \cdots x_{j_k} (j_1 < \cdots < j_k \leqq n-1)$ and C_P is generated by x_n and 1. Therefore the tensor product of two modules C_N and C_P is spanned by the elements $x_{j_1} \cdots x_{j_k}$ $(j_1 < \cdot < j_k \leqq n-1)$ and $x_{j_1} \cdot x_{j_k} x_n$, i.e., by $x_{i_1} \cdots x_{i_h} (i_1 < \cdots < i_h \leqq n)$, which proves our assertion.

5°. In particular when M has a finite base $x_1, \cdots, x_n$, A is a field of characteristic $\neq 2$ and $f=0$, the exterior algebra E over M is spanned by 2^n elements $x_{i_1} \cdots x_{i_h} (i_1 < \cdots < i_h)$. E is not only semi-graded, but also graded, and if we denote by $E = \sum_m E_m$ the decomposition into homogeneous components, E_m is spanned by the products of m elements $x_{i_1} \cdots x_{i_m} (i_1 < \cdots < i_m)$. Since $xx=0$ and $xy=-yx$ $(x, y \in M)$, we have $E_m=0$ if $m > n$, and E_n is spanned by only one element $x_1 \cdots x_n$. This proves that E_m is determined uniquely by M itself and does not depend upon the special choice of a base $x_1, \cdots, x_n$. Therefore if we take another *finite*[3] base $(y_1, \cdots, y_p)$ of M, we have $p=n$, i.e., the number of the elements of the base is invariant.

§ 4. **Canonical anti-automorphism.** The notations $A, M, f, \beta, T, \mathfrak{C}, C=T/\mathfrak{C}=C_+ + C_-$, π are all as before.

LEMMA 3.2. *To every linear mapping $\lambda: M \rightarrow A$, there exists a derivation d_λ in C of degree odd., i.e., $d_\lambda(C_+) \subset C_-$, and $d_\lambda(C_-) \subset C_+$, which satisfies*



(1) $\quad d_\lambda(\pi(x)) = \lambda(x) \cdot 1$ *for* $x \in M$,

and

(2) $\quad d_\lambda^2 = 0$.

PROOF. Since λ may be considered as a linear mapping $\lambda: T_1 \rightarrow T_0$, there exists a derivation δ_λ in T of degree -1 which extends λ, as we have proved in the previous chapter. We have

$$\delta_\lambda(x \otimes x - f(x) \cdot 1) = \delta_\lambda(x \otimes x) \qquad (\text{since } \delta_\lambda(1)=0)$$

3) If M has a finite base $x_1, \cdots, x_n$, this property holds if we delete the word "finite" for the base (y).

$$=\delta_\lambda(x)\otimes x - x\otimes\delta_\lambda(x) \qquad (\delta_\lambda \text{ is of degree -1})$$

$$=\lambda(x)\cdot 1\otimes x-\lambda(x)\cdot x\otimes 1=\lambda(x)(x-x)=0,$$

hence $\delta_\lambda(\mathfrak{C})=0$. Therefore δ_λ defines a derivation d_λ of C, which satisfies the condition (1). Also $\delta_\lambda{}^2$ is again a derivation since δ_λ is of odd degree, and we have

$$\delta_\lambda{}^2(x)=\delta_\lambda(\delta_\lambda(x))=\delta_\lambda(\lambda(x)\cdot 1)=\lambda(x)\cdot\delta_\lambda(1)=0,$$

which proves (2).

Now, if to an element $x(\neq 0)$ of M, there is a linear mapping $\lambda: M\to A$ such that $\lambda(x)\neq 0$, we obtain $d_\lambda(\pi(x))\neq 0$ and then $\pi(x)\neq 0$. When A is a field, every element $x(\neq 0)$ of M satisfies this condition, and we obtain

COROLLARY. *If A is a field, $\pi: M\to\pi(M)\subset C$ is an isomorphism, and we may identify M with $\pi(M)$ in C.*

Canonical anti-automorphism. Hereafter we assume that $\pi: M\to\pi(M)\subset C$ is an *isomorphism*. The above corollary asserts that this assumption holds when A is a field.

THEOREM 3.5. *There is an anti-automorphism on C of order 2, i.e., a mapping $u\to\bar{u}$ satisfying $\overline{uv}=\bar{v}\bar{u}$, which leaves the elements of M fixed.*

This mapping is called the *canonical* (*or main*) *anti-automorphism* of C.

PROOF. Let C' be the "opposite" of C, i.e., C' be a linear space with the same structure of A-module as C, and has a multiplication $u\times v=vu\,(u,v\in C)$. If $x\in M$, we have $x\times x-f(x)\cdot 1=xx-f(x)\cdot 1=0$ and then the injection of $M\to M\subset C'$ is extended to a homomorphism $C\ni u\to\bar{u}\in C$ by the universality of the tensor algebra. This homomorphism is linear and satisfies

$$\overline{uv}=\bar{u}\times\bar{v}=\bar{v}\bar{u} \tag{3}$$

and also $\bar{x}=x$, for $x\in M$. Taking the mapping $\bar{\ }$ again on (3), we have $\overline{\overline{uv}}=\overline{\bar{v}\bar{u}}=\bar{\bar{u}}\bar{\bar{v}}$ which proves that $u\to\bar{\bar{u}}$ is a homomorphism of $C\to C$. By $x=\bar{\bar{x}}\,(x\in M)$, $u\to\bar{\bar{u}}\,(u\in C)$ is the *identity*, and then $u\to\bar{u}$ is an involution. Hence $u\to\bar{u}$ is an isomorphism of C onto C', i.e., an anti-automorphism of C.

If $x_1, x_2, \cdots, x_h \in M$, we have

$$(4) \qquad \overline{x_1x_2 \cdots x_h} = \bar{x}_h \cdots \bar{x}_2\bar{x}_1 = x_h \cdot\cdot x_2x_1 .$$

When $f=0$ (the case of exterior algebra), we can interchange the right hand side of (4) by the anti-commutativity $xy=-yx$, and then we obtain

$$(4') \qquad \overline{x_1x_2 \cdot\cdot x_h} = (-1)^{(h-1)+(h-2)+\cdots+2+1} x_1x_2 \cdot\cdot x_h$$
$$= (-1)^{h(h-1)/2} x_1x_2 \cdot\cdot x_h$$

Now, since E_h is spanned by the elements $x_{i_1} \cdots x_{i_h}$, we have

$$(5) \qquad \bar{u} = (-1)^{h(h-1)/2} u \qquad \text{for all } u \in E_h .$$

In the case of exterior algebra, (5) is taken as the definition of the canonical anti-automorphism $\bar{u}$. We can prove directly that $u \to \bar{u}$ defined by (5) satisfies the conditions of the canonical anti-automorphism, using the property:

$$uv = (-1)^{hk} vu , \qquad \text{for } u \in E_h,\ v \in E_k .$$

§ 5. **Derivations in the exterior algebras; Trace.** In the case of an exterior algebra, we have the decomposition into homogeneous components $T=\sum_h T_h$, $E=\sum_h E_h$ in the Z-gradation.

LEMMA. 3.3. *If a linear mapping $\varphi : M \to E_h$ is given by a linear mapping $\psi : M \to T_h$, and the canonical mapping $\pi : T_h \to E_h$, we have a derivation d of degree $h-1$, which extends φ. d is uniquely determined*

The above condition on φ is always satisfied when M is a free module, or when A is a field, or when $h=1$ since $T_1=E_1$.

PROOF. The uniqueness follows from the fact that the derivation is determined uniquely by its effect on the generators.

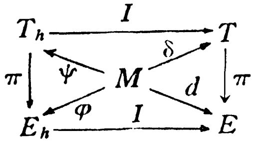
We shall prove the existence. Since $M \subset T_1$, we can take a derivation δ in T of degree $h-1$ which extends λ, by the considerations in the previous chapter. We have

$$(1) \qquad \delta(x \otimes x) = \delta(x) \otimes x + (-1)^{h-1} x \otimes \delta(x)$$
$$= \psi(x) \otimes x - (-1)^h x \otimes \psi(x)$$

and operating π on (1), we obtain

$$\pi(\delta(x \otimes x))=\varphi(x)\cdot x-(-1)^h x\cdot\varphi(x)=0$$

since $\varphi(x)\in E_h$, $x\in E_1$, and $\pi(x)=x$ for $x\in M$. Thus the ideal $\mathfrak{C}$ generated by $x\otimes x$ ($x\in M$) in T belongs to the kernel of π, and then δ defines a derivation d of E, which extends φ.



COROLLARY. *Any endomorphism $M\to M=E_1$ is extended to a uniquely determined derivatin of degree* 0 *in* E.

Now let $\mathfrak{F}(M)$ be the set of all endomorphisms of M. $\mathfrak{F}(M)$ is again a module over the basic ring A, and indeed it is also an algebra. For every element $\varphi\in\mathfrak{F}(M)$, we have a derivation d_φ of degree 0 by the above corollary.

LEMMA 3.4. *d_φ depends linearly on φ, i.e.,*

(2) $$d_{a\varphi+b\varphi'}=ad_\varphi+bd_{\varphi'} \quad (a, b\in A\,;\ \varphi, \varphi'\in\mathfrak{F}(M))\,.$$

and for the "bracket operation" $[\varphi, \varphi']=\varphi\varphi'-\varphi'\varphi$, d_φ satisfies

(3) $$d_{[\varphi,\varphi']}=[d_\varphi, d_{\varphi'}]\,(=d_\varphi d_{\varphi'}-d_{\varphi'}d_\varphi)$$

PROOF. Since (2) is proved similarly, we shall prove (3). The right hand side of (3) is again a derivation of E, since d_φ is of degree 0, it is sufficient to prove that both sides of (3) coincide with each other on the generator M of E. In fact, for $x\in M$, we have

$$\begin{aligned} d_{[\varphi,\varphi']}(x)&=[\varphi, \varphi'](x)=(\varphi\varphi'-\varphi'\varphi)(x)=\varphi\varphi'(x)-\varphi'\varphi(x)\\ &=d_\varphi(\varphi'(x))-d_{\varphi'}(\varphi(x))=d_\varphi d_{\varphi'}(x)-d_{\varphi'}d_\varphi(x)\\ &=(d_\varphi d_{\varphi'}-d_{\varphi'}d_\varphi)(x)\,, \end{aligned}$$

which proves our assertion.

Now we assume that E_n is a free module of rank 1 for some integer n, and $E_{n'}=\{0\}$ if $n'>n$. For example, this property holds if M is a free module with a base of n elements, i.e., when M is an n-dimensional vector space over A. Let ξ be a generator of E_n; $E_n=A\cdot\xi$. Since d_φ maps E_n into E_n, we have

$$d_\varphi\xi=s_\varphi\xi\,,$$

where s_φ is a uniquely determined element of A, which does not depend upon the special choice of ξ.

DEFINITION 3.4. *s_φ is called the trace of the endomorphism φ and is denoted by $s_\varphi = \mathrm{Tr}\ \varphi$.*

LEMMA 3.5. *$\mathrm{Tr}\ \varphi$ is linear in $\mathfrak{F}(M)$ and*

$$\mathrm{Tr}\ \varphi\varphi' = \mathrm{Tr}\ \varphi'\varphi. \tag{4}$$

PROOF. The former is evident from (2). For the latter, we have by definition,

$$d_\varphi d_{\varphi'}\xi = d_\varphi(s_{\varphi'}\xi) = s_{\varphi'}(d_\varphi \xi) = s_{\varphi'}s_\varphi \xi$$

and similarly

$$d_{\varphi'}d_\varphi \xi = s_\varphi s_{\varphi'}\xi\ .$$

But since we have assumed that A is *commutative*, we obtain

$$s_\varphi s_{\varphi'} = s_{\varphi'}s_\varphi\ ,$$

and therefore we have

$$\begin{aligned}(\mathrm{Tr}(\varphi\varphi' - \varphi'\varphi))\xi &= d_{\varphi\varphi'-\varphi'\varphi}\xi = (d_\varphi d_{\varphi'} - d_{\varphi'}d_\varphi)\xi \\ &= (s_{\varphi'}s_\varphi - s_\varphi s_{\varphi'})\xi = 0\end{aligned}$$

which proves (4).

REMARK. By (4) we have, for example,

$$\mathrm{Tr}\ \varphi\varphi'\varphi'' = \mathrm{Tr}\ \varphi''\varphi\varphi' = \mathrm{Tr}\ \varphi'\varphi''\varphi.$$

But an expression like $\mathrm{Tr}\ \varphi\varphi'\varphi'' = \mathrm{Tr}\ \varphi'\varphi\varphi''$ is *false* in general. Also $\mathrm{Tr}\,\varphi$ is not a homomorphism $\mathfrak{F}(M) \to A$.

When M is an n-dimensional vector space with a base $(x_1, \cdots, x_n)$, any element φ of $\mathfrak{F}(M)$ is represented by a square matrix (a_{ji}) of order n, such that

$$\varphi x_i = \sum_{j=1}^{n} a_{ji}x_j\ .$$

We shall show that the $\mathrm{Tr}\,\varphi$ defined above coincides with the classical one defined by the sum of the diagonal elements of a matrix. In our present case we have $E_n = Ax_1 \cdot x_n$, so we may take $\xi = x_1 \cdots x_n$. Then

$$\begin{aligned}(\mathrm{Tr}\,\varphi)\xi &= d_\varphi \xi = d_\varphi(x_1 \cdots x_n) \\ &= (d_\varphi x_1)x_2 \cdots x_n + x_1(d_\varphi x_2)x_3 \cdots x_n + \cdots + x_1 \cdots x_{n-1}(d_\varphi x_n)\end{aligned}$$

$$=\sum_{k=1}^{n} x_1 \cdots x_{k-1}\,\varphi(x_k)x_{k+1}\cdots x_n$$

$$=\sum_{k=1}^{n} x_1\cdots x_{k-1}\left(\sum_{i=1}^{n} a_{ik}x_i\right)x_{k+1}\cdots x_n\,,$$

since d_φ is of degree 0. But, since $xux=\pm xxu=0$, for $x \in M$, $u \in E$, we have

$$x_1\cdots x_{k-1}\left(\sum_{i=1}^{n} a_{ik}x_i\right)x_{k+1}\cdots x_n=a_{kk}\,x_1\cdots x_k\cdots x_n=a_{kk}\xi\,,$$

which proves that

$$(\mathrm{Tr}\,\varphi)\xi=\left(\sum_{k=1}^{n} a_{kk}\right)\xi\,,$$

i. e., $\mathrm{Tr}\,\varphi=a_{11}+a_{22}+\cdots+a_{nn}$.

Our definition of the trace seems to be intrinsic; it is evident from our definition that $\mathrm{Tr}\,\varphi$ is determined only by φ and does not depend upon the special choice of a base.

§ 6. **Orthogonal groups and spinors.** Let K be a field of characteristic $p(\geq 0)$, and V a finite dimensional vector space over K. Also let f be a quadratic form on V, β the associated bilinear form. We assume that β is *non-degenerate*, i. e., $\beta(x,y_0)=0$ for all $x \in V$, implies $y_0=0$. We denote by C the Clifford algebra associated to f over V.

DEFINITION 3.5. *An automorphism s of V is said to be orthogonal associated to f, if s leaves f invariant, i. e.,*

$$f(sx)=f(x) \qquad \textit{for all } x \in V.$$

We use the terminology "orthogonal transformation" instead of "orthogonal automorphism". The set of all orthogonal transformations forms a group which is called the *orthogonal group* of f and denoted by $O(f)$.

DEFINITION 3.6. *The set Γ of $u \in C$, such that u has an inverse u^{-1} and*

$$uVu^{-1} \subset V\,,\ \textit{i.e., } uxu^{-1} \in V \ \textit{ for all } x \in V\,,$$

forms a group, which is called the Clifford group of f.

If u belongs to the Clifford group Γ of f, $s_u(x)=uxu^{-1}$ is an orthogonal transformation, because

$$f(s_u(x))\cdot 1=(s_u(x))^2=(uxu^{-1})^2=ux^2u^{-1}=u(f(x)\cdot 1)u^{-1}=f(x)\cdot 1.$$

Hence the correspondence $\chi: u\to s_u$ is a linear representation of Γ, which is called the *vector representation* of Γ. The kernel of this representation is the set of invertible elements in the center of C.

If s is an automorphism of V, it is represented by a matrix and we have the determinant of s. If s is orthogonal, we have $\det s=\pm 1$. The set

$$\{s\in O(f)\mid \ \det s=1\}$$

forms a subgroup of $O(f)$, which is of index 2 unless the characteristic p of K is 2. When $p=2$, we have $\det s=1$ for all $s\in O(f)$.

Let $C=C_+ + C_-$ be the homogeneous decomposition of C in the semi-graded structure and we put $\Gamma^+=\Gamma\cap C_+$. We define $O^+(f)$ as follows:

(1)
$$\text{If } p\neq 2,\ O^+(f)=\{s\in O(f)\mid \det s=1\},$$
$$\text{If } p=2,\ O^+(f)=\{\chi(u)\mid u\in\Gamma^+\}.$$

Here we can prove that in both cases, $\{\chi(u)\mid u\in\Gamma^+\}$ *coincide with* $O^+(f)$, *and* $O^+(f)$ *is a subgroup of* $O(f)$ *with index* 2.

Let $u\to\bar{u}$ be the canonical anti-automorphism given in § 4. We can prove that $\bar{u}u\in K\cdot 1$ *for every* $u\in\Gamma^+$. Putting $\bar{u}u=\lambda(u)\cdot 1$, λ is a homomorphism $\Gamma\to K^*$, where K^* is the multiplicative group of non-zero elements in K. The kernel $\Gamma_0{}^+$ of this homomorphism λ is called the *reduced Clifford group*. Also we denote by Ω the image of $\Gamma_0{}^+$ under the vector representation χ, and call it the *reduced orthogonal group*.

When K is $\mathfrak{R}$, the real number field, and $f(x)=f\left(\sum_{i=1}^{n}\xi_i x_i\right)=\xi_1^2+\cdots+\xi_n^2$ (positive definite), $O^+(f)$ is the ordinary special orthogonal group. It is well known that $O^+(f)$ is not simply connected if $n\geqq 3$; the Poincaré group of $O^+(f)$ is actually of order 2 when $n\geqq 3$. Also we have $\Omega=O^+(f)$ and $\Gamma_0{}^+\to\Omega=O^+(f)$ is a covering mapping.

We now return to the general case. A linear subspace W of V is called *totally singular* if the restriction of the quadratic form on W is the zero quadratic form over W. All maximal totally singular subspaces of V have the same dimension, and the common dimension is called the *index* of f. It is evident that f is of index 0 if and only if there is no $x \neq 0$ with $f(x)=0$. Here we have the following results:

If the index of f is not 0, we have

$$O^+(f)/\Omega \cong K^*/(K^*)^2 .\text{[4]} \tag{2}$$

Ω is the commutator subgroup of $O(f)$ except when K has only two elements, dim $V=4$ *and f is of index* 2. *If furthermore $n=$* dim $V \geqq 3$, *Ω is the commutator subgroup of $O^+(f)$. Also when $n=$*dim $V=2$, *$O^+(f)$ is abelian, and its commutator group consists only of $\{e\}$.*

On the other hand, the structure of Ω when the index of f is 0 is quite unknown.

Now we assume that V is of even dimension, namely $2n$, and let $x_1,\cdots, x_n, y_1,\cdots, y_n$ be the base of V. Suppose that f can be reduced to the following form:

$$f\left(\sum \xi_i x_i+\sum \eta_i y_i\right)=\sum \xi_i \eta_i . \tag{3}$$

When K is algebraically closed, every quadratic form whose β is non-degenerate can be reduced to this form. On the contrary, if K is not algebraically closed, such reduction is not always possible, as will be shown by an example of $\xi^2+\eta^2$ over the real number field. Under these assumptions, the Clifford algebra C is isomorphic to a full matric algebra and has the dimension 2^{2n}, while C_+ is of dimension 2^n. There is a minimal left ideal $\mathfrak{A}$ in C, of dimension 2^n. For $u \in C$, we have $\xi \in \mathfrak{A} \rightarrow u\xi \in \mathfrak{A}$ and then the transformation $\lambda_u : \xi \rightarrow u\xi$ is a representation of C. λ_u induces a representation over $\Gamma^+ (\subset C)$, and is an isomorphic representation on Γ^+. This is called the *spin representation* of Γ^+, and the elements of $\mathfrak{A}$ are called *spinors*.

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4) K^* means the multiplicative group of elements $\neq 0$ in K.

The origin of this name is as follows. When E. Cartan classified the simple representations of all simple Lie algebras, he discovered a new representation of the orthogonal Lie algebra. But he dit not give a specific name to it, and far later, he called the elements on which this new representation operates *spinors*, generalizing the terminology adopted by the physicists in a special case for the rotational group of the three dimensional space.

The spin representation of Γ is simple except when K has only two elements, $n=1$ and f is of index 1. Also the spin representation of Γ^+ is either simple or the sum of two simple representations.

We may assume further that $\mathfrak{A}$ is *homogeneous in the semi-graded structure of* C, i. e.,

$$(4) \qquad \mathfrak{A}=\mathfrak{A}_+ + \mathfrak{A}_- , \text{ where } \mathfrak{A}_\pm=\mathfrak{A} \cap C_\pm .$$

This corresponds to the decomposition of the spin representation into two irreducible ones, and each of them is called the *half spin representation*. Each half spin representation is of degree 2^{n-1}.

When $n>2$, the kernel of each half spin representation is of order 1 or 2. On the contrary, if $n=2$, i. e., if V is of dimension 4, it is not so. This corresponds to the fact that the rotational group of dimension 4 is not simple. When $n=2$, let Δ_1, Δ_2 be the kernels of the two half spin representations of $\Gamma_0{}^+$; we have

$$\Gamma_0{}^+=\Delta_1 \cdot \Delta_2 \qquad \text{(direct product)},$$

and the spin representation of $\Gamma_0{}^+$ splits into two parts. One of them operates on $\mathfrak{A}_+$ and leaves invariant $\mathfrak{A}_-$, while the other operates on $\mathfrak{A}_-$ and fixes $\mathfrak{A}_+$. Moreover the covering group of the orthogonal group splits into the product of two subgroups. The representation λ_u $(u \in \Delta_1)$ produces all automorphisms of determinant 1 on $\mathfrak{A}_+$, and then each of Δ_1 and Δ_2 is isomorphic to the multiplicative group of 2-2-matrices of determinant 1.

Now let $\Gamma_0{}'^+$ be the reduced Clifford group of a quadratic form in 3 variables. We have $\Gamma_0{}'^+ \subset \Gamma_0{}^+$, and $\Gamma_0{}'^+$ is imbedded into both Δ_1 and Δ_2. Also $\Gamma_0{}'^+ \rightarrow \Delta_1$ is an isomorphism onto. The Clifford group which covers the orthogonal group is isomorphic to Δ_1, and this corresponds to the *D-part* of spinors so called by the physicists.

When K is $\mathfrak{R}$, the real number field, a quadratic form cannot always be written in the form (3) as we have remarked above. But if we extend K to the complex number field, the representation as (3) is possible, and the real quadratic form f is extended to a Hermitian form, while the representation $\Gamma_0'^{+} \to \Delta_1$ is given by a unitary matrix. This may be an answer to the question why the spinors are treated on the complex number field.

CHAPTER IV. SOME APPLICATIONS OF EXTERIOR ALGEBRAS.

§ 1. **Plücker coordinates.** Let K be a field, V a finite n-dimensional vector space over K, and E the exterior algebra over V. The decomposition into homogeneous components of E is denoted by $E=\sum_m E_m$. If $x_1,\cdots,x_n$ is the base of V, the $\binom{n}{m}$ elements $x_{i_1}\cdots x_{i_m}(i_1<\cdots<i_m)$ form a base of E_m.

DEFINITION 4.1. *An element a of E_m is called decomposable if a is the product of m elements of V.*

Any element in E_m is the sum of a finite number of decomposable elements. We remark that $aa=0$ if a is decomposable.

Let W be an m-dimensional linear subspace of V with a base $y_1,\cdots,y_m$. By the canonical mapping of W into V, the exterior algebra F of W is naturally isomorphic to the subalgebra of E generated by W, and the homogeneous component F_m of degree m in F is therefore in E_m. On the other hand, F_m is of dimension 1, spanned by $y_1\cdots y_m$ Thus to any linear subspace W in V of dimension m, there corresponds a 1-dimensional subspace of E_m, namely F_m. Conversely, if F_m is a 1-dimensional subspace of E_m spanned by a decomposable element, we have an m-dimensional linear subspace W, such that the homogeneous component of degree m of the exterior algebra over W is F_m. Also we have $xF_m=0$, if and only if $x\in W$. In fact, let $y_1,\cdots,y_m$ be a base of W. If $x\in W$, we may take $x=y_1$, and by $F_m=K\{y_1\cdots y_m\}$ we have $xy_1\cdots y_m=0$, and then $xF_m=0$. Conversely, if $x\notin W$, the $m+1$ elements $x, y_1,\cdots,y_m$ being linearly independent, they are part of the base of V, which proves $xy_1\cdots y_m\neq 0$. Also we have

THEOREM 4.1. *The elements $x_1,\cdots,x_m$ of V are linearly independent if and only if $x_1\cdots x_m\neq 0$ in E.*

Also the family of all m-dimensional linear subspaces of V, and the family of 1-dimensional subspaces of E_m which are spanned by decomposable elements, correspond in a one-to-one manner with each other. If we take a base $(x_1, \ldots, x_n)$ of V, we have

$$y_1 \cdots y_m = \sum_{i_1 < \cdots < i_m} \alpha_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}, \quad \alpha_{i_1 \cdots i_m} \in K$$

for a base $y_1, \cdots, y_m$ of W. The ratios of various $\alpha_{i_1 \cdots i_m}$'s are invariant if we take another base $y_1', \cdots, y_m'$ of W, since $y_1 \cdots y_m$ is a base of F_m.

DEFINITION 4.2. *These ratios of $\alpha_{i_1 \cdots i_m}$'s are called the Plücker coordinates* of W.

Since the base of F_m is decomposable, the Plücker coordinates can not freely be chosen, but must satisfy some identities. For example, if $n=4$ and $m=2$, the identity reads:

$$\alpha_{12}\alpha_{34} + \alpha_{31}\alpha_{24} + \alpha_{23}\alpha_{14} = 0.$$

§ 2. **Exponential mapping.** Let V be a vector space (not necessarily finite dimensional) over the field K, and E be the exterior algebra of V. We shall define the exponential mapping in E. The ordinary exponential function is defined by the power series

$$\exp x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^m}{m!} + \cdots. \tag{1}$$

For $x \in E$, we may consider the multiplication in E for $x^2, x^3, \cdots$, and if x is a homogeneous element of degree >0, we have $x^m = 0$ for sufficiently large m. But it will cause a difficulty to define $\exp x$ by (1), because of the factor $\frac{1}{m!}$, unless the characteristic of K is 0. So, we shall proceed in another way. If x is decomposable, we have $x^2=0$ and then $\exp x$ may be defined simply by $1+x$. If we restrict ourselves to elements $a, b, \cdots$ of even degree, we have the commutativity $ab=ba$, and we may expect the "addition theorem" of exponential function:

$$\exp(a+b) = (\exp a)(\exp b). \tag{2}$$

Hence $\exp x$ may be defined through decomposing x into a sum of decomposable elements. However, in order to assert the uniqueness of this definition, we shall begin with proving some lemmas.

LEMMA 4.1. *If $x \in E_h$, $h \geq 1$, $x \neq 0$, then there exist h derivations $d_1, \cdots, d_h$ of degree -1 such that $d_1 \cdots d_h(x) \neq 0$.*

Since K is a field, we may say that $d_1 \cdots d_h(x)=1$ in multiplying by a suitable scalar.

PROOF. Let $(y_i)_{i \in I}$ be a base of V; introduce a relation of linear order into I. A binary relation like $i_\mu < i_\nu$ means always the relation with respect to this order. Since the elements $y_{i_1} \cdots y_{i_h} (i_1 < \cdots < i_h)$ form a base of E_h, we can write

$$x = \sum_{i_1 < \cdots < i_h} \alpha(i_1, \cdots, i_h) y_{i_1} \cdots y_{i_h}, \quad \alpha(i_1, \cdots, i_h) \in K. \tag{3}$$

Since $x \neq 0$, there is at least a sequence of indices $(\bar{i}_1, \cdots, \bar{i}_h)$ such that $\alpha(\bar{i}_1, \cdots, \bar{i}_h) \neq 0$. Now for each $\nu=1, \cdots, h$, there exists a linear function λ_ν in V such that

$$\lambda_\nu(y_{\bar{i}_\nu})=1 \quad \text{and} \quad \lambda_\nu(y_i)=0 \text{ for all } i \neq \bar{i}_\nu. \tag{4}$$

By the extension theorem, there is a derivation d_ν of degree -1 which extends λ_ν. We have by the definition of the derivation,

$$d_\nu(y_{i_1} \cdots y_{i_h})=(d_\nu(y_{i_1}))y_{i_2} \cdots y_{i_h} - y_{i_1}(d_\nu(y_{i_2}))y_{i_3} \cdots y_{i_h} + - \cdots$$
$$+(-1)^{h-1} y_{i_1} \cdots y_{i_{h-1}}(d_\nu(y_{i_h})).$$

But (4) shows that $d_\nu(y_i)=\lambda_\nu(y_i) \neq 0$ only if $i=\bar{i}_\nu$, and then we obtain

$$d_\nu(y_{i_1} \cdots y_{i_h})=0 \quad \text{if} \quad \bar{i}_\nu \notin \{i_1, \cdots, i_h\},$$

When $i_\nu \in \{i_1, \cdots, i_h\}$, namely $i_\nu = i_r$, we have

$$d_\nu(y_{i_1} \cdots y_{i_h})=(-1)^{r-1} y_{i_1} \cdots \hat{y}_{i_r} \cdots y_{i_h},$$

where the symbol $\wedge$ above y_{i_r} means that this factor should be omitted from the product. Then we have

$$d_h(x)=\sum \pm \alpha(i_1, \cdots, i_h) y_{i_1} \cdots \hat{y}_{\bar{i}_h} \cdots y_{i_h},$$

where the summation is taken over the family of indices such that

$$i_1 < \cdots < i_h, \qquad \bar{i}_h \in \{i_1, \cdots, i_h\}.$$

By successive applications of d_ν, we have

$$d_1 \cdots d_h(x)=\pm \alpha(\bar{i}_1, \cdots, \bar{i}_h),$$

since the terms in the right hand side of (3) vanish unless $(i_1,\cdots,i_h)$ contains all $\bar{i}_1,\cdots,\bar{i}_h$. This proves our assertion since we have assumed that

$$\alpha(\bar{i}_1,\cdots,\bar{i}_h)\neq 0.$$

LEMMA 4.2. *An element $x\in E$ has the property that $d(x)=0$ for every homogeneous derivation d of degree -1 of E, if and only if $x\in E_0$.*

PROOF. It is evident that $x\in E_0$ implies $d(x)=0$ for every derivation d of degree -1. For the converse, we shall prove the contraposition, i.e., the proposition that if $x\notin E_0$, then there exists a derivation of degree -1 such that $d(x)\neq 0$. Let $x=\sum_h x_h$ be the homogeneous decomposition of x. Since $x\notin E_0$, we have an integer $h\geq 1$ such that $x_1=\cdots=x_{h-1}=0$, $x_h\neq 0$. By the above Lemma 4.1, we have a derivation d of degree -1, such that $d(x_h)\neq 0$. Since $d(x_0)=0$, and $d(x)=d(x_h)+d(x_{h+1})+\cdots$ is the homogeneous decomposition of $d(x)$, we have $d(x)\neq 0$ from $d(x_h)\neq 0$, which proves our statement.

LEMMA 4.3. *If a is decomposable of degree ≥ 2, and d is a derivation of degree -1, we have $ad(a)=0$.*

PROOF. Putting $a=xb$, where $x\in V$ and b is again a decomposable element of degree ≥ 1, we have $d(a)=d(x)b-xd(b)$, and then

$$ad(a)=xbd(x)b-xbxd(b)=d(x)xbb\pm xxbd(b)=0$$

since $xx=0$, $bb=0$.

If the degree of a is even and the characteristic of K is not 2, this lemma can also be proved from $d(aa)=0$.

LEMMA 4.4. *Let $a_1,\cdots,a_k$ be decomposable elements of strictly positive even degree, such that $a_1+\cdots+a_k=0$. Then we have*

$$\sum_{i_1<\cdots<i_m} a_{i_1}a_{i_2}\cdots a_{i_m}=0. \tag{5}$$

PROOF. We first remark that the case of $m=2$ is easily proved unless the characteristic of K is 2. In fact, we have $a_i^2=0$, and $a_ia_j=a_ja_i$, because the a_i's are decomposable elements of even degree. Hence we obtain

$$0=(a_1+\cdots+a_k)^2=\sum_i a_i^2+\sum_{i,j} a_i a_j=2\sum_{i<j} a_i a_j,$$

and then the constant factor 2 can be removed, provided that the characteristic is not 2.

But we shall give a demonstration which is valid in the general cases. Putting

$$u=\sum_{i_1<\cdots<i_m} a_{i_1}\cdots a_{i_m},$$

it is sufficient to show that $d(u)=0$ for every derivation d of degree -1 by Lemma 4.2. Since a_i's are all of even degree, they are commutative with any element in E. Thus we have

$$d(u)=\sum_{i_1<\cdots<i_m}[(d(a_{i_1}))a_{i_2}\cdots a_{i_m}+a_{i_1}(d(a_{i_2}))a_{i_3}\cdots a_{i_m}+$$
$$\cdots+a_{i_1}\cdots a_{i_{m-1}}(d(a_{i_m}))]$$
$$=\sum_{i_1<\cdots<i_m}[a_{i_2}\cdots a_{i_m}(d(a_{i_1}))+a_{i_1}a_{i_3}\cdots a_{i_m}(d(a_{i_2}))+$$
$$\cdots+a_{i_1}\cdots a_{i_{m-1}}(d(a_{i_m}))]$$
$$=\sum_{\substack{j_1<\cdots<j_{m-1}\\ i\neq j_1,\ldots,j_{m-1}}} a_{j_1}\cdots a_{j_{m-1}}d(a_i)$$
$$=\Big(\sum_{j_1<\cdots<j_{m-1}} a_{j_1}\cdots a_{j_{m-1}}\Big)\Big(\sum_{i=1}^{k} d(a_i)\Big)-\sum_{\substack{j_1<\cdots<j_{m-1}\\ i\in\{j_1,\ldots,j_{m-1}\}}} a_{j_1}\cdots a_{j_{m-1}}d(a_i).$$

But since $\sum d(a_i)=d(\sum a_i)=0$ by our assumption, and $a_i d(a_i)=0$ by Lemma 4.3, we have $d(u)=0$ which proves our statement.

Now we shall give the definition of the exponential mapping on the space F of elements whose degrees are even:

$$F=E_2+E_4+\cdots+E_{2h}+\cdots.$$

First we define $\exp a=1+a$ if a is *decomposable*. For any element $u\in F$, it is possible in at least one way to represent u in the form $u=a_1+\cdots+a_k$ where each a_i is decomposable and of even degree, because each E_{2h} has a base consisting of decomposable elements. Then we define

$$\exp u=(1+a_1)(1+a_2)\cdots(1+a_k). \tag{6}$$

While the representation $u=a_1+\cdots+a_k$ by decomposable elements is not unique, exp u is determined uniquely by u. Precisely speaking, if we represent u in two manners

$$u=a_1+\cdots+a_k=b_1+\cdots+b_l,$$

where a_i and b_j are decomposable, we have

$$(1+a_1)(1+a_2)\cdots(1+a_k)=(1+b_1)(1+b_2)\cdots(1+b_l). \tag{7}$$

In fact, putting $a_{k+1}=-b_1, \ldots, a_{k+l}=-b_l$, we have $a_1+a_2+\cdots+a_{k+l}=0$, where $a_1,\ldots,a_{k+l}$ are all decomposable. Then we have by Lemma 4.4 that

$$\sum_{i_1<\cdots<i_m} a_{i_1}\cdots a_{i_m}=0. \tag{5}$$

The expression $(1+a_1)(1+a_2)\cdots(1+a_{k+l})$ is expanded by the "polynomial theorem" since a_i's are mutually commutative, and all terms except 1 vanish because of (5). Thus we obtain,

$$\begin{aligned}(1+a_1)(1+a_2)\cdots(1+a_{k+l})\\ =(1+a_1)\cdots(1+a_k)(1-b_1)\cdots(1-b_l)=1.\end{aligned} \tag{8}$$

On the other hand we have $(1+b_j)(1-b_j)=1-b_j^2=1$, since b_j is decomposable. Multiplying $(1+b_1)(1+b_2)\cdots(1+b_l)$ to both sides of (8), we have (7), since a_i, b_j are mutually commutative.

DEFINITION 4.3. *The mapping $u\to$ exp u defined above is called the exponential mapping of $F\to E$.*

It is evident from the definition that exp u satisfies

$$\exp(a+b)=(\exp a)(\exp b) \quad (a, b\in F). \tag{2'}$$

In particular when V is a finite dimensional vector space, whose dimension is even, namely $2m$, we take a base $y_1,\cdots,y_{2m}$. Let Γ be a homogeneous element of degree 2. The homogeneous component of degree 2 of exp Γ is a multiple of $y_1\cdots y_{2m}$, namely

$$(\exp \Gamma)_{2m}=P_\Gamma(y_1\cdots y_{2m}), \quad P_\Gamma\in K.$$

DEFINITION 4.4. *P_Γ is called the Pfaffian of $\Gamma\in E_2$.*

If Γ is represented by a sum of m decomposable elements[1] of

1) This condition can be proved to be satisfied by the theory of skew-symmetric forms, but here we assume this property.

degree 2, putting $\Gamma=a_1+\cdots+a_m$, we have

$$\exp \Gamma=(1+a_1)\cdots(1+a_m),$$

and expanding the right hand side by polynomial theorem, the term of degree $2m$ is merely $a_1\cdots a_m$. On the other hand, using the polynomial theorem for $\Gamma^m=(a_1+\cdots+a_m)^m$, and noting that $a_i^2=0$, we have $\Gamma^m=m!\, a_1\cdots a_m$, which proves

(9) $$m!\,(\exp \Gamma)_{2m}=\Gamma^m.$$

If the characteristic of K is 0 or relatively prime to $m!$, we obtain

(9′) $$(\exp \Gamma)_{2m}=\Gamma^m/m!.$$

§ 3. **Determinants.** Let V be a finite n-dimensional vector space over K. An endomorphism s of V is extended uniquely to a homomorphism S_s of $E\to E$, which is homogeneous of degree 0. Since E_n is of dimension 1 and $S_s(E_n)\subset E_n$, there exists a uniquely determined scalar Δ_s such that

(1) $$S_s z=\Delta_s z \qquad \text{for} \qquad z\in E_n.$$

DEFINITION 4.5. *This Δ_s is called the determinant of the endomorphism s and denoted by $\Delta_s=\det s$.*

The properties of determinant are easily proved from this definition. For example, we shall show

THEOREM 4.2. 1° $(\det s)(\det s')=\det(s\circ s')$.

2°. $\det s\neq 0$ *if and only if s is an automorphism of V.*

PROOF. 1°. Let s, s' be two endomorphisms of V. $S_s\circ S_{s'}$ is a homomorphism of $E\to E$ which coincide with $S_{s\circ s'}$ in V, and thus we have $S_s\circ S_{s'}=S_{s\circ s'}$. Therefore, for $z\in E_n$, we obtain

$$\Delta_{s\circ s'}z=S_{s\circ s'}z=S_s\circ S_{s'}z=S_s(\Delta_{s'}z)=\Delta_{s'}(S_s z)=\Delta_{s'}\Delta_s z,$$

which proves our assertion, since we have assumed that K is commutative.

2°. If $(x_1,\cdots,x_n)$ is a base of V, E_n is spanned by $x_1\cdots x_n$, and we have

(2) $$\Delta_s(x_1\cdots x_n)=S_s(x_1\cdots x_n)$$
$$=S_s(x_1)\cdots S_s(x_n)=s(x_1)\cdots s(x_n)$$

since S_s is a homomorphism. Therefore by Theorem 4.1, det $s=0$ if and only if $s(x_1)\cdots s(x_n)$ are linearly dependent, and then it is equivalent to the fact that s is not an automorphism of V.

Now, if we write

$$s(x_i)=\sum_{j=1}^{n} a_{ji}\, x_j\,,$$

we have

$$\begin{aligned}\Delta_s(x_1\cdots x_n)&=s(x_1)\cdots s(x_n)=(\sum a_{j1}x_j)\cdots(\sum a_{jn}x_j)\\&=\sum_{i_1,\dots,i_n} a_{i_1 1}\cdots a_{i_n n}\,(x_{i_1}\dots x_{i_n})\,.\end{aligned}$$

But $x_{i_1}\cdots x_{i_n}=0$ if there exists a pair of indices such that $i_\mu=i_\nu$ $(\mu\neq\nu)$, and when the $(i_1,\cdots,i_n)$ are all distinct, we have $x_{i_1}\cdots x_{i_n}=$ sgn $(i_1,\cdots,i_n)\,(x_1\cdots x_n)$, where sgn $(i_1,\cdots,i_n)$ is $+1$ or -1 according as $(i_1,\cdots,i_n)$ is an even or odd permutation of $(1,\cdots,n)$. Thus we obtain

$$\Delta_s(x_1\cdots x_n)=\sum_{i_1,\cdots,i_n} a_{i_1 1}\cdots a_{i_n n}\ \mathrm{sgn}\,(i_1,\cdots,i_n)\ (x_1\cdots x_n)$$

which proves that

(3) $$\det s=\det(a_{ji})=\sum \mathrm{sgn}\,(i_1,\cdots,i_n)\,a_{i_1 1}\cdots a_{i_n n}\,,$$

where the summation is taken in all the sets $(i_1,\cdots,i_n)$ such that $i_1,\cdots,i_n$ are all distinct. This shows that det s may be expressed as a polynomial with the coefficients ±1 in the a_{ji}'s.

Now, let U be a vector space of $2n$ dimensions over K; we assume that U is given by the direct sum of two n-dimensional linear subspaces V and W: $U=V+W$. Let $(x_1,\cdots,x_n)$, $(y_1,\cdots,y_n)$ be the bases of V and W respectively. Taking together with x_i's and y_i's, they form a base of U. We define a bilinear form $\beta(x,y)$ on $U\times U$ in setting

(4) $$\beta(x_i,x_j)=\beta(y_i,y_j)=0\,,\ \beta(x_i,y_j)=\delta_{ij}\ \ (i,j=1,\cdots,n)\,,$$

then β is a symmetric non-degenerate bilinear form on $U\times U$, satisfying $\beta(V,V)=\beta(W,W)=0$.

The set of all linear functionals over V is again an n-dimensional vector space over K which is called the *dual space* of V and denoted by V^*. In our present case, to any $y \in W$, the functional over V defined by

$$(5) \qquad \lambda_y(x)=\beta(x, y) \quad \text{for} \quad x \in V,$$

is linear, and belongs to V^*. Since $\lambda_{y_j}(x_i)=\delta_{ij}$, the mapping λ: $y \to \lambda_y$ is a linear isomorphism of W onto V^*. Therefore we may identify W and V^* with each other.

If s is an automorphism of V, we can define an automorphism ${}^ts$ of V^* by

$$({}^ts\,\lambda)\,(x)=\lambda(sx).$$

We have easily $({}^ts)^{-1}={}^t(s^{-1})$ and this automorphism of V^* is denoted by $\hat{s}$. Since V^* is identified with W, $\hat{s}$ is also an automorphism of W. Then there exists an automorphism S_s of U which coincides with s on V and $\hat{s}$ on W respectively. We shall prove the following:

THEOREM 4.3. *We have* $\det S_s=1$.

We first prove the following:

LEMMA 4.5. *We put*

$$\Theta=\sum_{i=1}^{n} x_i \otimes y_i\,,$$

which is an element of degree 2 in the tensor algebra over U. S_s extends to an automorphism of the tensor algebra over U, and this extended automorphism leaves Θ fixed.

Proof of Lemma 4.5. What we have to prove is the identity

$$(6) \qquad \sum_{i=1}^{n} sx_i \otimes \hat{s}y_i=\sum_{i=1}^{n} x_i \otimes y_i\,.$$

Since we have identified V^* with W, putting

$$sx_i=\sum_{k=1}^{n} a_{ki}x_k\,,$$

we have by (4) and (5)

$$\beta(x_i, {}^tsy_k)=({}^ts\lambda_{y_k})\,(x_i)=\lambda_{y_k}(sx_i)=\beta(sx_i, y_k)$$

$$=a_{ki}=\beta(x_i, \sum_{j=1}^{n} a_{kj}y_j)\,.$$

This implies

$$(7) \qquad {}^t s y_k = \sum_{j=1}^{n} a_{kj} y_j ,$$

which proves that the matrix corresponding to ${}^t s$ is the *transposed matrix* of the automorphism s. Applying $\hat{s}$ on (7), we have

$$y_k = \sum_{i=1}^{n} a_{ki} (\hat{s} y_i) ,$$

and then

$$\sum_{i=1}^{n} s x_i \otimes \hat{s} y_i = \sum_{i,k} a_{ki} x_k \otimes \hat{s} y_i = \sum_{k=1}^{n} \left(x_k \otimes \sum_{i=1}^{n} a_{ki} (\hat{s} y_i) \right)$$

$$= \sum_{k=1}^{n} x_k \otimes y_k ,$$

which proves (6).

Now we return to the proof of $\det S_s = 1$. Since the exterior algebra E_U over U was defined by the canonical image of the tensor algebra over U (see § 2, Chap. III), we denote the canonical image of Θ in E_U by Γ. Γ is represented by

$$\Gamma = \sum_{i=1}^{n} x_i y_i .$$

By Lemma 4.5, the automorphism 早[2)] of E_U which extends S_s leaves Γ fixed. Then 早 leaves ($\exp \Gamma$) invariant, because the exponential mapping is defined intrisically in the exterior algebra. More precisely, since $x_i y_i$ and 早$(x_i y_i) = s(x_i)\hat{s}(y_i)$ are decomposable, we have

$$\begin{aligned} \exp \Gamma &= (1 + x_1 y_1)(1 + x_2 y_2) \cdots (1 + x_n y_n) \\ &= (1 + \text{早}(x_1 y_1))(1 + \text{早}(x_2 y_2)) \cdots (1 + \text{早}(x_n y_n)) \\ &= \text{早}((1 + x_1 y_1)(1 + x_2 y_2) \cdots (1 + x_n y_n)) = \text{早}(\exp \Gamma) \end{aligned}$$

Hence 早 leaves also invariant the component $(\exp \Gamma)_{2n}$ of the highest dimension of $\exp \Gamma$. On the other hand, Γ being the sum of n decomposable elements, we have

$$(\exp \Gamma)_{2n} = x_1 y_1 x_2 y_2 \cdots x_n y_n ,$$

as we remarked at the end of § 2, and it is a basic element $\neq 0$ in $(E_U)_{2n}$. Therefore we have by the definition of the determinant

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2) Read "sô".

$$(\det S_s)(x_1 y_1 \cdots x_n y_n) = \mathfrak{S}(x_1 y_1 \cdots x_n y_n)$$
$$= x_1 y_1 \cdots x_n y_n \neq 0,$$

which proves $\det S_s = 1$.

THEOREM 4.4. *Let U, V, W be as before. If s is an automorphism of U, which leaves V and W fixed, and if we denote by s_V, s_W the contractions of s into V and W respectively, then*

$$\det s = (\det s_V)(\det s_W).$$

PROOF. This theorem follows from $E_U \cong E_V \otimes E_W$, but we shall give a simpler demonstration. Let $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ be bases of V and W respectively. We denote by $\mathfrak{S}$ the automorphism of E_U which extends s. By the definition of the determinant, we have

$$\mathfrak{S}(x_1 \cdots x_n) = (\det s_V)(x_1 \cdots x_n)$$

since E_V is generated by $x_1, \cdots, x_n$ in E_U and $\mathfrak{S}(E_V) \subset E_V$. Similarly we have

$$\mathfrak{S}(y_1 \cdots y_n) = (\det s_W)(y_1 \cdots y_n),$$

and then

$$(\det s)(x_1 \cdots x_n y_1 \cdots y_n) = \mathfrak{S}(x_1 \cdots x_n y_1 \cdots y_n)$$
$$= \mathfrak{S}(x_1 \cdots x_n)\mathfrak{S}(y_1 \cdots y_n) = (\det s_V)(x_1 \cdots x_n)(\det s_W)(y_1 \cdots y_n)$$
$$= (\det s_V)(\det s_W)(x_1 \cdots x_n y_1 \cdots y_n),$$

which proves our statement.

COROLLARY. *The determinant of a matrix s is equal to the determinant of its transposed one*: $\det {}^t s = \det s$.

PROOF. The automorphism S_s of U which coincides with s and $\hat{s}$ on V and W respectively satisfies the conditions of Theorem 4.4. Then we have, from two theorems given above, that

$$(\det s)(\det \hat{s}) = \det S_s = 1.$$

On the other hand $(\det \hat{s})(\det {}^t s) = 1$, because of $\hat{s} = ({}^t s)^{-1}$, which proves our assertion.

§ 4. **An application to combinatorial topology.** As an application of the theory of exterior algebra, we shall give a demonstration of a fundamental property in the theory of combinatorial topology:

that the boundary of a boundary is 0.

We are now dealing with the combinatorial topology, and take all vertices P_α. In the singular homology theory, all points in the space are the $\{P_\alpha\}$. We construct a vector space V of which the P_α's form its base. Any element of V is a 0-dimensional chain in the homology theory. Now a simplex σ is ordinarily defined as a set of a finite number of P_α's: $\sigma=(P_{\alpha_1},\cdots,P_{\alpha_h})$ with an orientation which makes σ skew-symmetric symbol. This law of orientation is quite the same one as in the exterior algebra; it is appropriate to represent the simplex $\sigma=(P_{\alpha_1},\cdot\cdot,P_{\alpha_h})$ by the element $P_{\alpha_1}\cdots P_{\alpha_h}$ in the exterior algebra E_V over V. A p-dimensional simplex is of degree $p+1$ in E_V. Next we define the boundary operation. There exists a linear mapping δ on V such that $\delta P_\alpha=1$ for all α. Then we have a derivation d of degree -1 which extends δ. If we apply d to a simplex $\sigma=(P_{\alpha_1},\cdot,P_{\alpha_h})$, we have

$$d\sigma=(dP_{\alpha_1})P_{\alpha_2}\cdots P_{\alpha_h}-P_{\alpha_1}(dP_{\alpha_2})P_{\alpha_3}\cdots P_{\alpha_h}$$

$$+-\cdots\pm P_{\alpha_1}\cdots P_{\alpha_{h-1}}(dP_{\alpha_h})$$

$$=P_{\alpha_2}\ldots P_{\alpha_h}-P_{\alpha_1}P_{\alpha_3}\ldots P_{\alpha_h}+-\cdots+(-1)^{r-1}P_{\alpha_1}\cdots\hat{P}_{\alpha_r}\cdots P_{\alpha_h}$$

$$+-\cdots+(-1)^{h-1}P_{\alpha_1}\cdots P_{\alpha_{h-1}}.$$

This expression coincides with the ordinary definition of the boundary operation. So, we *define the boundary operation by d.* Then d being a derivation of *odd* degree, d^2 is again a derivation and the property

$$d^2(P_\alpha)=d(dP_\alpha)=d(1)=0,$$

proves $d^2=0$. Hence the boundary of a boundary is 0.

Although there are many other interesting applications of the exterior algebras, we omit them because of the restriction of time. We only mention an application to physics; the equaions of Maxwell in the theory of electro-magnetism may be represented elegantly using the forms of exterior algebra.[3)]

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3) See Erich Kähler, Bemerkungen über die Maxwellschen Gleichungen, Hamburg Abhandlungen, 12 (1938), pp. 1-28.

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Editor's Note

The foregoing is the reproduction of the beautiful lectures delivered by Professor C. Chevalley at the University of Tokyo in April- June 1954, after the notes taken by S. Hitotumatu and N. Yoneda. S. Fukutomi has prepared the manuscript for pr'nting, and all three have read the proofs. The Editor is responsible for any mistakes in the text.